

Exact Persistence Exponent for the 2D-Diffusion Equation and Related Kac Polynomials

Mihail Poplavskyi¹ and Grégory Schehr²

¹King's College London, Department of Mathematics, London WC2R 2LS, United Kingdom

²LPTMS, CNRS, Univ. Paris-Sud, Université Paris-Saclay, 91405 Orsay, France



(Received 3 July 2018; published 12 October 2018)

We compute the persistence for the 2D-diffusion equation with random initial condition, i.e., the probability $p_0(t)$ that the diffusion field, at a given point \mathbf{x} in the plane, has not changed sign up to time t . For large t , we show that $p_0(t) \sim t^{-\theta(2)}$ with $\theta(2) = 3/16$. Using the connection between the 2D-diffusion equation and Kac random polynomials, we show that the probability $q_0(n)$ that Kac's polynomials, of (even) degree n , have no real root decays, for large n , as $q_0(n) \sim n^{-3/4}$. We obtain this result by using yet another connection with the truncated orthogonal ensemble of random matrices. This allows us to compute various properties of the zero crossings of the diffusing field, equivalently of the real roots of Kac's polynomials. Finally, we unveil a precise connection with a fourth model: the semi-infinite Ising spin chain with Glauber dynamics at zero temperature.

DOI: 10.1103/PhysRevLett.121.150601

Persistence and first-passage properties have attracted a lot of interest during the last decades in physics, both theoretically [1–3] and experimentally [4–7], as well as in mathematics [8]. For a stochastic process $X(t)$, the persistence $P_0(t)$ is the probability that it has not changed sign up to time t . In nonequilibrium statistical physics, this is an interesting observable which is nonlocal in time and thus carries useful information on the full history of the system on a given time interval [9].

In many physically relevant situations, $P_0(t)$ decays algebraically at late time $t \gg 1$, $P_0(t) \sim t^{-\theta}$, where θ is called the *persistence exponent* [1–3]. For instance, for Brownian motion or Lévy flights, which are Markov processes, $\theta = 1/2$. But in many cases, in particular, for coarsening dynamics [10], and more generally for non-Markov processes, the exponent θ is nontrivial and extremely hard to compute [1–3]. Consequently, there are very few non-Markov processes, for which θ is known exactly. One notable example is the 1D Ising chain with Glauber dynamics. In this case, at temperature $T = 0$, the persistence exponent for the local magnetization can be computed exactly, yielding $\theta_{\text{Ising}} = 3/8$ [11,12].

Another example that has attracted a lot of attention [13–19] is the d -dimensional diffusion equation where the scalar field $\phi(\mathbf{x}, t)$ at point $\mathbf{x} \in \mathbb{R}^d$ and time t evolves as $\partial_t \phi(\mathbf{x}, t) = \Delta \phi(\mathbf{x}, t)$, where initially $\phi(\mathbf{x}, t=0)$ is a Gaussian random field, with zero mean and short-range correlations $\langle \phi(\mathbf{x}, 0) \phi(\mathbf{x}', 0) \rangle = \delta^d(\mathbf{x} - \mathbf{x}')$. For a system of linear size L , the persistence $p_0(t, L)$ is the probability that $\phi(\mathbf{x}, t)$, at some fixed point \mathbf{x} in space, does not change sign up to time t [13,14]. We assume that \mathbf{x} is far enough from the boundary, where the system is invariant under translations, and $p_0(t, L)$ is thus independent of \mathbf{x} . It was

shown [13,14] that $p_0(t, L)$ takes the scaling form, for large t and large L , with t/L^2 fixed

$$p_0(t, L) \sim L^{-2\theta(d)} h(L^2/t), \quad (1)$$

with $h(u) \rightarrow c_1$, a constant, when $u \rightarrow 0$ and $h(u) \propto u^{\theta(d)}$ when $u \rightarrow \infty$, where $\theta(d)$ was found, numerically, to be nontrivial, e.g., $\theta(1) = 0.12050(5)\dots$, $\theta(2) = 0.1875(1)\dots$ [13,14,16]. Remarkably, the persistence for $d = 1$ was observed in experiments on magnetization of spin polarized Xe gas and the exponent $\theta_{\text{exp}}(1) \simeq 0.12$ was measured [7]. This scaling form (1) shows that $p_0(t, L) \sim t^{-\theta(d)}$ for an infinite system. Alternatively, $\theta(d)$ can also be obtained, in a finite system of size L , from $p_0(t, L) \sim L^{-2\theta(d)}$ for $t \gg L^2$. To study $p_0(t, L)$ it is useful to introduce the normalized process $X(t) = \phi(\mathbf{0}, t)/\langle \phi(\mathbf{0}, t)^2 \rangle$ [20]. Being Gaussian, $X(t)$ is completely characterized by its autocorrelation function which, for an infinite system $L \rightarrow \infty$, behaves like $C(t, t') = \langle X(t)X(t') \rangle = [2\sqrt{tt'}/(t+t')]^{d/2}$. In terms of logarithmic time $T = \ln t$, $Y(T) = X(e^T)$ is a Gaussian stationary process with covariance $c(T) = [\text{sech}(T/2)]^{d/2}$ (see Fig. 1). In particular, $c(T) \approx 1 - dT^2/16$ for $T \rightarrow 0$, indicating a smooth process with a finite density of zero crossings $\rho_0 = (2\pi)^{-1} \sqrt{d/2}$ [21]. Although several very accurate approximation schemes exist to compute $\theta(d)$ [13–15,22,23], there is not a single value of d for which this persistence exponent could be computed exactly. In this Letter, we focus on the case $d = 2$. As we show below, this case is particularly interesting because it is related to a variety of other interesting models (see Fig. 1), in particular to the celebrated Kac's random polynomials [17–19,24]. These are polynomials of degree n

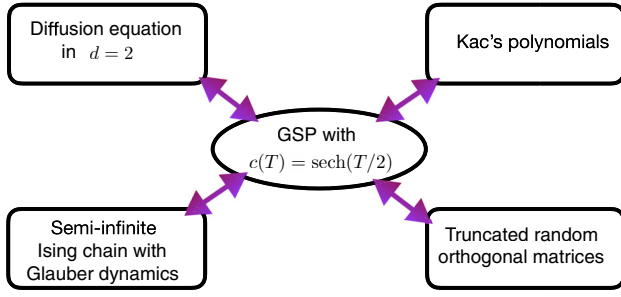


FIG. 1. Connections between the four models studied here: they are related to the same Gaussian stationary process (GSP) with correlator $c(T) = \text{sech}(T/2)$. Our main result is the exact value of the persistence exponent for this GSP, $\theta(2) = b = 3/16$, together with the full statistics of its zero crossings (5)–(7).

$$K_n(x) = \sum_{i=0}^n a_i x^i, \quad (2)$$

where the coefficients a_i 's are independent and identically distributed (i.i.d.) real Gaussian random variables of zero mean and unit variance. Of course, $K_n(x)$ has n roots in the complex plane. These roots tend to cluster, when $n \rightarrow \infty$, close to the unit circle centered at 0. But because the coefficients a_i 's are real, the statistics of the number of real roots is singular, and it has thus generated a lot of interest [25,26]. In particular, the average number of real roots grows, for $n \gg 1$, like $\sim (2/\pi) \ln n$, hence much smaller than n . It is thus natural to ask: what is the probability $q_0(n)$ that $K_n(x)$ has no real roots for an even n ? It was shown in Ref. [24] that $q_0(n)$ decays to zero as $q_0(n) \sim n^{-4b}$ where, remarkably, b turns out to be the persistence exponent for the diffusion equation in $d = 2$ [17,18], i.e., $b = \theta(2)$.

To establish the connection between these two problems, one first notices that almost all the real roots of $K_n(x)$ lie very close to ± 1 , in a window of size $\mathcal{O}(1/n)$ [27]. In addition, it was shown in Ref. [24] (see also Ref. [28]) that, for large n , the real roots of $K_n(x)$ behave independently and identically within each of the four subintervals $(-\infty, -1]$, $[-1, 0]$, $[0, 1]$, and $[1, +\infty)$. One can thus focus on one of these intervals, say $[0, 1]$, and consider $\tilde{q}_0(x, n)$, which is the probability that K_n has no real root in $[0, x]$, with $0 < x \leq 1$. Clearly, $q_0(n) \sim [\tilde{q}_0(1, n)]^4$ for large n . For $x \rightarrow 1^-$, it was shown in Refs. [17–19,24] that the behavior of $\tilde{q}_0(x, n)$ is governed by the zero-crossings properties of the GSP $Y(T)$ with covariance $c(T) = \text{sech}(T/2)$, i.e., the same GSP that governs the zero crossings of the 2D-diffusion equation (see Fig. 1). In particular, in the scaling limit $n \rightarrow \infty$, $x \rightarrow 1^-$ with $n(1-x)$ fixed [recall that the scaling region around ± 1 is of order $\mathcal{O}(1/n)$], one can show that $\tilde{q}_0(x, n)$ takes the scaling form [17,18],

$$\tilde{q}_0(x, n) \sim n^{-b} \tilde{h}[n(1-x)], \quad (3)$$

with $\tilde{h}(u) \rightarrow c_2$, a constant, for $u \rightarrow 0$ and $\tilde{h}(u) \sim u^b$ for $u \rightarrow \infty$. The large u behavior follows from the fact that $\tilde{q}_0(x, n \rightarrow \infty)$ is well defined. This form (3) is the exact analogue of the finite size scaling form in Eq. (1), with n playing the role of L^2 and $(1-x)$ the role of inverse time $1/t$ [17,18]. This implies that $b = \theta(2)$ can be extracted either for finite n , from $\tilde{q}_0(1, n) \sim n^{-b}$, or for $n \rightarrow \infty$ (i.e., for the Gaussian power series) from $\tilde{q}_0(x, n \rightarrow \infty) \sim (1-x)^b$, as $x \rightarrow 1^-$. The study of this exponent b has generated a lot of interest in the math literature [8,19,24,29–31] and the best existing bounds are $0.144338\dots = 1/(4\sqrt{3}) \leq b \leq 1/4 = 0.25$ [29,31].

Main results.—Here, we exploit a connection between the Kac's polynomials and the so-called truncated real orthogonal ensemble of random matrices [32–34] (see below) to obtain the exact result

$$b = \theta(2) = 3/16 = 0.1875, \quad (4)$$

which is fully consistent with numerical simulations [13,14,16] and the above exact bounds [29,31] as well as with a recent conjecture in number theory [35]. We also compute the full probability distribution of the number of zero crossings N_t of $\phi(\mathbf{0}, t)$ up to time t . Let $p_k(t, L) = \text{Prob}(N_t = k)$ and $p_k(t) = p_k(t, L \rightarrow \infty)$. We show that, for large t and k , with $k/\ln t$ fixed, $p_k(t)$ takes the large deviation form proposed in Refs. [17,18],

$$p_k(t) \sim t^{-\varphi(k/\ln t)}, \quad (5)$$

where the large deviation function $\varphi(x)$ is computed exactly. Its asymptotic behaviors are given by

$$\varphi(x) \sim \begin{cases} \frac{3}{16} + x \ln x, & x \rightarrow 0 \\ \frac{1}{2\sigma^2} (x - \frac{1}{2\pi})^2, & |x - \frac{1}{2\pi}| \ll 1 \\ \frac{\pi^2}{4} x^2 - \frac{\ln 2}{2} x, & x \rightarrow \infty, \end{cases} \quad (6)$$

with $\sigma^2 = 1/\pi - 2/\pi^2$. Close to the center, for $x = 1/(2\pi)$, the quadratic behavior in Eq. (6) shows that $p_k(t)$ has a Gaussian peak, of width $\sigma \ln(t)$, close to its maximum $\langle N_t \rangle \approx \ln(t)/(2\pi)$. However, away from this central Gaussian regime, $p_k(t)$ is flanked, on both sides of $\langle N_t \rangle$, by nontrivial tails (6)—the right one being, however, still Gaussian (at leading order), though different from the Gaussian central part. Finally, we also obtain the large t behavior of the cumulants $\langle N_t^p \rangle_c$ of arbitrary order p

$$\langle N_t^p \rangle_c \sim \kappa_p \ln t, \quad \kappa_p = \frac{2^{p-2}}{\pi^2} \sum_{m=1}^p (-2)^{m-1} \Gamma^2\left(\frac{m}{2}\right) \mathcal{S}_p^{(m)}, \quad (7)$$

where $\mathcal{S}_p^{(m)}$ is the Stirling number of the second kind [36]. In particular, one recovers $\kappa_1 = 1/(2\pi)$ and $\kappa_2 = \sigma^2 = 1/\pi - 2/\pi^2$ (see Ref. [22]) and obtains, for instance,

$\kappa_3 = 4/\pi - 12/\pi^2$. Our main results in Eqs. (4), (6), and (7) are not only relevant for the 2D-diffusion equation, but also for the whole class of models discussed in this Letter that can be mapped onto the GSP, $Y(T)$, with correlator $c(T) = \text{sech}(T/2)$ (see Fig. 1). In particular, the probability $\mathcal{P}_k(T)$ that it has exactly k zeros up to T is given, for large T and $k = \mathcal{O}(T)$, by $\mathcal{P}_k(T) \sim e^{-T\varphi(k/T)}$, with the same function $\varphi(x)$ (6). Similarly, the cumulants of the number of zero crossings are given by Eq. (7), with the substitution $\ln t \rightarrow T$ and the same coefficients κ_p . We further show that this GSP has a Pfaffian structure: the multitime correlation functions of $\text{sgn}[Y(T)]$ can be written as Pfaffians [37]. Besides, we demonstrate that the zeros of $Y(T)$ form a Pfaffian point process [37]. Finally, we establish an exact mapping between the 2D-diffusing field and the semi-infinite Ising spin chain with Glauber dynamics at zero temperature. As we will see, using the exact result for the persistence exponent of the full chain, $\theta_{\text{Ising}} = 3/8$ [11,12], this connection provides an alternative derivation of the exact result $\theta(2) = b = \theta_{\text{Ising}}/2 = 3/16$.

Truncated random orthogonal matrices.—We consider the set of real orthogonal matrices, of size $(2n+1) \times (2n+1)$ (with n a positive integer), uniformly distributed, with the Haar measure, on the orthogonal group $O(2n+1)$. Let O be such a real random orthogonal matrix, such that $OO^T = \mathbb{I}$. We define its truncation M_{2n} as the $2n \times 2n$ random matrix obtained by removing the last column and row from the matrix O

$$O = \begin{pmatrix} M_{2n} & \mathbf{u} \\ \mathbf{v}^T & a \end{pmatrix}, \quad (8)$$

where \mathbf{u}, \mathbf{v} are column vectors and a is a scalar. Such truncated matrices, together with their unitary counterpart, were studied in the context of mesoscopic physics [34,42] and extreme statistics [43]. The orthogonality condition $OO^T = \mathbb{I}$ implies that $M_{2n}M_{2n}^T = \mathbb{I} - \mathbf{u}\mathbf{u}^T$ and, hence, all the eigenvalues of M_{2n} lie in the unit disk (since their norm is less than unity). They are the roots of the characteristic polynomial $g_M(z) = \det(z\mathbb{I} - M_{2n})$, which after some manipulations, can be written as [44] [Lemma 6.7.2] (see also [37])

$$g_M(z) = \det O \det(zM_{2n} - \mathbb{I})(a + z\mathbf{v}^T(\mathbb{I} - zM_{2n})^{-1}\mathbf{u}). \quad (9)$$

Since the eigenvalues z_i 's of M_{2n} are such that $0 < |z_i| < 1$, one has necessarily that $\det(z_i M_{2n} - \mathbb{I}) = z_i^N g_M(1/z_i) \neq 0$ in the right-hand side of Eq. (9). This implies that the z_i 's are the zeros of $[a + z\mathbf{v}^T(\mathbb{I} - zM_{2n})^{-1}\mathbf{u}]$ [see Eq. (9)]. Expanding in powers of z shows that the eigenvalues of M_{2n} are the zeroes of the series

$$F_{2n}(z) = a + \sum_{k=1}^{\infty} z^k \mathbf{v}^T M_{2n}^{k-1} \mathbf{u}, \quad (10)$$

with $|z| < 1$ (note that $\mathbf{v}^T M_{2n}^{k-1} \mathbf{u}$ are real numbers). Quite remarkably, one can show [44] that the scaled sequence of the real coefficients of the series in Eq. (10), i.e., $\sqrt{2n}\{a, \mathbf{v}^T \mathbf{u}, \mathbf{v}^T M_{2n} \mathbf{u}, \mathbf{v}^T M_{2n}^2 \mathbf{u}, \dots\}$, converges, as $n \rightarrow \infty$, to a sequence of i.i.d. Gaussian random variables, with zero mean and unit variance. This implies that, for $n \rightarrow \infty$, the real eigenvalues of M_{2n} in Eq. (8) and the real zeroes of $K_n(x)$ in Eq. (2) in the interval $[-1, 1]$ share the same statistics.

But what about the connection between these two models for finite n ? In fact, it is known that the real eigenvalues of M_{2n} accumulate close to $x = \pm 1$, also on a window of size $\mathcal{O}(1/n)$ [34], like for the Kac's polynomials [27]. Hence, if one considers the probability $\tilde{Q}_0(x, n)$ that M_{2n} has no real eigenvalue in $[0, x]$, it is natural to expect that, as for Kac's polynomials (3), for large n and $x \rightarrow 1$ keeping $n(1-x)$ fixed, $\tilde{Q}_0(x, n)$ behaves as

$$\tilde{Q}_0(x, n) \sim n^{-\gamma} \tilde{H}(n(1-x)), \quad (11)$$

where the exponent γ is yet unknown and, *a priori*, the scaling function $\tilde{H}(u)$ is different for $\tilde{h}(u)$ in Eq. (3). However, for $n \rightarrow \infty$, we have seen that $\tilde{q}_0(x, n)$ and $\tilde{Q}_0(x, n)$ do coincide, since they both correspond to the probability that the (infinite) Gaussian power series has no real root in $[0, x]$. This implies that $\tilde{Q}_0(x, n \rightarrow \infty) = \tilde{q}_0(x, n \rightarrow \infty) \sim (1-x)^b$, which, together with the scaling form (11), shows that $\gamma = b$. Finally, since we expect that $\tilde{Q}_0(1, n)$ exists, one has $\tilde{H}(u) \rightarrow c_3$, a constant, when $u \rightarrow 0$, and therefore $\tilde{Q}_0(1, n) \sim n^{-b}$ for large n . One can also consider the probability $Q_0(x, n)$ that M_{2n} has no real eigenvalue in $[-x, x]$. Using the statistical independence of the positive and negative real eigenvalues for large n , one has $Q_0(x, n) \sim [\tilde{Q}_0(x, n)]^2$, and in particular $Q_0(1, n) \sim n^{-2b}$ for large n . Using similar arguments, one can show that the full statistics of the zero-crossings of the diffusion equation (equivalently of the real roots of $K_n(x)$) can be obtained, at leading order for large n , from the statistics of the number of real eigenvalues \mathcal{N}_n of the random matrix M_{2n} , which we now study. Our analysis follows the line developed in Ref. [45] where the real eigenvalues of real Ginibre matrices were studied.

We start with the full joint distribution of the eigenvalues of M_{2n} (8). Since M_{2n} is real and of even size $2n$, it has l (with l even) real eigenvalues (and possibly $l = 0$), denoted by $\lambda_1 \leq \dots \leq \lambda_l$, and $m = n - l/2$ pairs of complex conjugate eigenvalues $z_1 = x_1 + iy_1, z_2 = \bar{z}_1, \dots, z_{2m-1} = x_m + iy_m, z_{2m} = \bar{z}_{2m-1}$ with $x_1 \leq \dots \leq x_m$. Then the ordered eigenvalues of M_{2n} conditioned to have l real eigenvalues have the joint distribution [33,34]

$$p^{(l,m)}(\vec{\lambda}, \vec{z}) = C |\Delta(\vec{\lambda}, \vec{z})| \prod_{j=1}^l w(\lambda_j) \prod_{j=1}^{2m} w(z_j), \quad (12)$$

where $C \equiv C_{m,n}$ is a normalization constant,

$$w^2(z) = (2\pi|1 - z^2|)^{-1}, \quad (13)$$

and Δ is a Vandermonde determinant. The generating function (GF) of \mathcal{N}_n (the number of real roots) reads

$$\langle e^{s\mathcal{N}_n} \rangle_{M_{2n}} = \left\langle \prod_{i=1}^{2n} 1 - (1 - e^s) \chi_{\mathbb{R}}(\zeta_i) \right\rangle_{M_{2n}}, \quad (14)$$

for $s < 0$, where the product runs over all the eigenvalues ζ_i 's (both real and complex) of M_{2n} . In Eq. (14), $\chi_{\mathbb{R}}(z) = 1$ if z is real and 0 otherwise and $\langle \cdots \rangle_{M_{2n}}$ denotes an average over the joint distribution (12), further summed over all possible (l, m) [46,47]. It turns out that such averages (14) can be computed explicitly in terms of Pfaffians [46–52], as follows. Let $f(z)$ be any smooth integrable complex function, and $\{p_j(z)\}_{j=0}^{2n-1}$ be an arbitrary sequence of monic polynomials of degree j , then

$$\left\langle \prod_{i=1}^{2n} f(\zeta_i) \right\rangle_{M_{2n}} = \frac{\text{Pf}(U_f)}{\text{Pf}(U_1)}, \quad (15)$$

where Pf denotes a Pfaffian [53] and U_f is a skew symmetric (i.e., antisymmetric) matrix of size $2n \times 2n$ with entries $u_{j,k} = (p_{j-1}f, p_{k-1}f)_w$ and skew product

$$(h, g)_w = \int_{\mathbb{R}^2} h(x)g(y) \text{sgn}(y-x)w(x)w(y)dx dy + 2i \int_{\mathbb{C}} h(z)g(\bar{z}) \text{sgn}[\text{Im}(z)]w(z)w(\bar{z})d^2z, \quad (16)$$

where $w(z)$ is given in Eq. (13). To compute the ratio in Eq. (15), it is convenient to choose the monic polynomials $p_j(z)$ to be skew orthogonal with respect to the product (16) [with this choice, the denominator in Eq. (15) is easy to compute [37]]. Using these polynomials [33], the GF in Eq. (14) can be evaluated explicitly using Eq. (15), leading to [54] (see also [55])

$$\langle e^{s\mathcal{N}_n} \rangle_{M_N} = \det_{0 \leq j, k \leq n-1} \left(\delta_{j,k} - \frac{1 - e^{2s}}{\pi(j+k+1/2)} \right). \quad (17)$$

Let us denote by H_n the $n \times n$ matrix with entries $h_{j,k} = [\pi(j+k+1/2)]^{-1}$. We write the determinant in Eq. (17) as $\det(\mathbb{I} - \alpha H_n) = \exp \text{Tr}[\ln(\mathbb{I} - \alpha H_n)]$, with $\alpha = (1 - e^{2s})$ and then expand the logarithm, to get $\det(\mathbb{I} - \alpha H_n) = \exp[\sum_{m \geq 1} (\alpha^m/m) \text{Tr}(H_n^m)]$. The asymptotic analysis of the traces yields [56]

$$\text{Tr} H_n^m = \frac{1}{2\pi} \int_0^\infty \text{sech}^m\left(\frac{\pi u}{2}\right) du \log n [1 + o(1)], \quad n \rightarrow \infty.$$

By summing up these traces, we obtain

$$\langle e^{s\mathcal{N}_n} \rangle_{M_{2n}} = n^{[1/(2\pi)] \int_0^\infty \log\{1 - (1 - e^{2s}) \text{sech}[(\pi u)/2]\} du + o(1)}. \quad (18)$$

For $s < 0$, the integral can be calculated explicitly as

$$\langle e^{s\mathcal{N}_n} \rangle_{M_{2n}} \sim n^{\psi(s)}, \quad \psi(s) = \frac{1}{8} - \left[\frac{\sqrt{2}}{\pi} \cos^{-1}\left(\frac{e^s}{\sqrt{2}}\right) \right]^2. \quad (19)$$

By taking $s \rightarrow -\infty$ we get the probability that M_{2n} has no real eigenvalues, using $Q_0(1, n) = \text{Prob}(\mathcal{N}_n = 0) = \lim_{s \rightarrow -\infty} \langle e^{s\mathcal{N}_n} \rangle_{M_{2n}} \sim n^{-2b}$. From Eq. (19), we thus obtain $b = (-1/2) \lim_{s \rightarrow -\infty} \psi(s) = 3/16$, as announced in Eq. (4). From the GF in Eq. (19), we also obtain the cumulants of \mathcal{N}_n . To export these results to the 2D-diffusion equation, we recall that the number of zero crossings N_t identifies with the positive real eigenvalues \mathcal{N}_n^+ of M_{2n} . Hence, for $n \gg 1$, the number of positive and negative \mathcal{N}_n^\pm real eigenvalues are both independent and identically distributed [54], one obtains that $\langle e^{s\mathcal{N}_n^+} \rangle_{M_{2n}} \sim n^{\psi(s)/2}$. By further expanding $\psi(s)$ close to $s = 0$ [37], one finally obtains the result announced in Eq. (7). Similarly, transposing this result $\langle e^{s\mathcal{N}_n^+} \rangle_{M_{2n}} \sim n^{\psi(s)/2}$ to the diffusion equation, one obtains the large deviation form in Eq. (5) with $\varphi(x) = \max_{s \in \mathbb{R}} [sx - \psi(s)/2]$ [57]. From this relation, together with the expression for $\psi(s)$ in Eq. (19), we obtain the asymptotic behaviours given in Eq. (6) [37].

Several results found so far point to an intriguing connection with the zero temperature Glauber dynamics of the Ising spin chain [11,12]. First, $b = 3/16$ is thus half of the persistence exponent, $\theta_{\text{Ising}} = 3/8$, found there [11,12]. In fact, $3/16$ is exactly the persistence exponent corresponding to the spin at the origin of the semi-infinite Ising chain [11,12]. Furthermore, the expression found for $\psi(s)$ in Eq. (19) is strongly reminiscent of the expression found for the persistence exponent for the q -state Potts chain, with $T = 0$ Glauber dynamics [see, e.g., Eq. (2) of Ref. [11]]. So what is this connection?

To understand it, let us come back to the 2D-diffusion field $X(t) = \phi(\mathbf{0}, t)/\langle \phi(\mathbf{0}, t)^2 \rangle$ and consider the “clipped” process $\text{sgn}[X(t)]$ [58]. As recalled above $X(t)$ has the same statistical properties as the Kac’s polynomials $K_n(x)$ in the limit $n \rightarrow \infty$ and $x \rightarrow 1$. Transposing recent results obtained for Kac’s polynomials in the limit $n \rightarrow \infty$ [59], we can compute the multitime correlation functions of $\text{sgn}[X(t)]$, which are given by Pfaffians [53]

$$\langle \text{sgn}[X(t_1)] \cdots \text{sgn}[X(t_m)] \rangle \sim \text{Pf}(A) \quad (20)$$

for $1 \ll t_1 \ll t_2 \ll \dots \ll t_{2m}$, and where $A = (a_{i,j})_{1 \leq i,j \leq 2m}$ is a $2m \times 2m$ antisymmetric matrix with $a_{i,i} = 0$ and for $i < j$, $a_{i,j} = -a_{j,i}$ where

$$a_{i,j} = \frac{2}{\pi} \sin^{-1} [\langle X(t_i) X(t_j) \rangle] = \frac{2}{\pi} \sin^{-1} \left(\frac{2\sqrt{t_i t_j}}{t_i + t_j} \right). \quad (21)$$

By symmetry, the even correlation functions vanish. For $m = 1$, Eqs. (20) and (21) hold for any normalized Gaussian process. However, for $m > 1$, this Pfaffian structure, which holds for the GSP $Y(T) = X(e^T)$, is nontrivial.

Let us now consider the semi-infinite Ising spin chain, whose configuration at time t is given by $\{\sigma_i(t)\}_{i \geq 0}$, with $\sigma_i(t) = \pm 1$. Initially, $\sigma_i(0) = \pm 1$ with equal probability $1/2$ and, at subsequent time, the system evolves according to the Glauber dynamics at $T = 0$ (see Refs. [11,12] for details). Using the formulation of the dynamics in terms of coalescing random walks [11,12], we show that the multitime correlation functions of σ_0 are also given by the same Pfaffian formula (20), namely, for $1 \ll t_1 \ll t_2 \ll \dots \ll t_{2m}$ [37]

$$\langle \sigma_0(t_1) \dots \sigma_0(t_{2m}) \rangle \sim \text{Pf}(A), \quad (22)$$

with precisely the same antisymmetric matrix A (21). Therefore, we conclude that $\text{sgn}[X(t)]$ for the 2D-diffusion equation and $\sigma_0(t)$ in the semi-infinite Ising chain with Glauber dynamics are actually the same process in the large time limit [60]. One can then use the known result for the persistence of $\sigma_0(t)$ [11,12] to conclude that $b = 3/16$, as found above by a completely different method. Note that the exact relation found here $\sigma_0(t) \propto \text{sgn}[\phi(\mathbf{0}, t)]$ (for $t \gg 1$), where $\phi(\mathbf{x}, t)$ is the 2D-diffusing field, is reminiscent, albeit different from, the so-called OJK approximate theory [61] in phase ordering kinetics [62], which instead approximates the 1D spin field by the sign of the 1D diffusing field.

To conclude, we have computed exactly the persistence exponent of 2D-diffusion equation, or equivalently the one of Kac's polynomials, $\theta(2) = b = 3/16$. This was done in two different ways: (i) by using the connection to truncated random orthogonal matrices, and for which our results are actually mathematically rigorous [54], (ii) by establishing an exact mapping to the semi-infinite Ising chain with Glauber dynamics at $T = 0$ (see Fig. 1). Thanks to (i), we computed the full statistics of the number of the zero crossings (5)–(7). These RMT tools will certainly be useful to compute other properties of the GSP with correlator $c(T) = \text{sech}(T/2)$ and of the different physical models associated with it (see Fig. 1).

G.S. wishes to thank warmly S.N. Majumdar for illuminating discussions and ongoing collaborations on this topic and related ones. We would also like to acknowledge A. Dembo, I. Dornic, Z. Kabluchko, M. Krishnapur,

and P. Le Doussal for useful discussions and comments. We also thank the Les Houches School of Physics, where this project was initiated. This work was partially supported by EPSRC EP/N009436/1 as well as by the ANR Grant No. ANR-17-CE30-0027-01 RaMaTraF.

- [1] S.N. Majumdar, Persistence in nonequilibrium systems, *Curr. Sci.* **77**, 370 (1999).
- [2] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, England, 2001).
- [3] A.J. Bray, S.N. Majumdar, and G. Schehr, Persistence and first-passage properties in non-equilibrium systems, *Adv. Phys.* **62**, 225 (2013).
- [4] W.Y. Tam, R. Zeitak, K.Y. Szeto, and J. Stavans, First-Passage Exponent in Two-Dimensional Soap Froth, *Phys. Rev. Lett.* **78**, 1588 (1997).
- [5] D.B. Dougherty, I. Lyubintsky, E.D. Williams, M. Constantin, C. Dasgupta, and S.D. Sarma, Experimental Persistence Probability for Fluctuating Steps, *Phys. Rev. Lett.* **89**, 136102 (2002).
- [6] K.A. Takeuchi and M. Sano, Evidence for geometry-dependent universal fluctuations of the Kardar-Parisi-Zhang interfaces in liquid-crystal turbulence, *J. Stat. Phys.* **147**, 853 (2012).
- [7] G.P. Wong, R.W. Mair, R.L. Walsworth, and D.G. Cory, Measurement of Persistence in 1D Diffusion, *Phys. Rev. Lett.* **86**, 4156 (2001).
- [8] F. Aurzada and T. Simon, Persistence Probabilities and Exponents, in *Lévy matters V. Lecture Notes in Mathematics*, (Springer, Cham, 2015), Vol. 2149, pp. 183–224.
- [9] A. Watson, Persistence pays off in defining history of diffusion, *Science* **274**, 919 (1996).
- [10] B. Derrida, A.J. Bray, and C. Godrèche, Nontrivial exponents in the zero-temperature dynamics of the 1d Ising and Potts models, *J. Phys. A* **27**, L357 (1994).
- [11] B. Derrida, V. Hakim, and V. Pasquier, Exact First-Passage Exponents of 1d Domain Growth—Relation to a Reaction-Diffusion Model, *Phys. Rev. Lett.* **75**, 751 (1995).
- [12] B. Derrida, V. Hakim, and V. Pasquier, Exact exponent for the number of persistent spins in the zero-temperature dynamics of the One-dimensional Potts model, *J. Stat. Phys.* **85**, 763 (1996).
- [13] S.N. Majumdar, C. Sire, A.J. Bray, and S.J. Cornell, Nontrivial Exponent for Simple Diffusion, *Phys. Rev. Lett.* **77**, 2867 (1996).
- [14] B. Derrida, V. Hakim, and R. Zeitak, Persistent Spins in the Linear Diffusion Approximation of Phase Ordering and Zeros of Stationary Gaussian Processes, *Phys. Rev. Lett.* **77**, 2871 (1996).
- [15] H.J. Hilhorst, Persistence exponent of the diffusion equation in ϵ dimensions, *Physica (Amsterdam)* **277A**, 124 (2000).
- [16] T.J. Newman and W. Loinaz, Critical Dimensions of the Diffusion Equation, *Phys. Rev. Lett.* **86**, 2712 (2001).
- [17] G. Schehr and S.N. Majumdar, Statistics of the Number of Zero Crossings: from Random Polynomials to Diffusion Equation, *Phys. Rev. Lett.* **99**, 060603 (2007).

- [18] G. Schehr and S.N. Majumdar, Real roots of random polynomials and zero crossing properties of diffusion equation, *J. Stat. Phys.* **132**, 235 (2008).
- [19] A. Dembo and S. Mukherjee, No zero-crossings for random polynomials and the heat equation, *Ann. Probab.* **43**, 85 (2015).
- [20] One could choose any other point $\mathbf{x} \neq \mathbf{0}$ and define $X(t)$ in terms of $\phi(\mathbf{x}, t)$ since, for $L \rightarrow \infty$, the system is invariant under space translations.
- [21] S.O. Rice, Mathematical analysis of random noise, *Bell Syst. Tech. J.* **23**, 282 (1944).
- [22] S.N. Majumdar and A.J. Bray, Persistence with Partial Survival, *Phys. Rev. Lett.* **81**, 2626 (1998).
- [23] G.C.M.A. Ehrhardt, S.N. Majumdar, and A.J. Bray, Persistence exponents and the statistics of crossings and occupation times for Gaussian stationary processes, *Phys. Rev. E* **69**, 016106 (2004).
- [24] A. Dembo, B. Poonen, Q.M. Shao, and O. Zeitouni, Random polynomials having few or no real zeros, *J. Am. Math. Soc.* **15**, 857 (2002).
- [25] For a review, see A. Edelman, and E. Kostlan, How many zeros of a random polynomial are real?, *Bull. Am. Math. Soc.* **32**, 1 (1995), and references therein.
- [26] P. Bleher and X. Di, Correlations between zeros of a random polynomial, *J. Stat. Phys.* **88**, 269 (1997).
- [27] A.P. Aldous and Y.V. Fyodorov, Real roots of random polynomials: universality close to accumulation points, *J. Phys. A* **37**, 1231 (2004).
- [28] It can also be seen by computing the correlation function of the density of real zeros between different points belonging to two distinct subintervals and observe that it decays exponentially for large n [18].
- [29] W.V. Li and Q.M. Shao, A normal comparison inequality and its applications, *Probab. Theory Relat. Fields* **122**, 494 (2002).
- [30] W.V. Li and Q.M. Shao, Recent developments on lower tail probabilities for Gaussian processes, *Cosmos* **1**, 95 (2005).
- [31] G. Molchan, Survival exponents for some Gaussian processes, *Int. J. Stochastic Anal.* **2012**, 137271 (2012).
- [32] M. Krishnapur, From random matrices to random analytic functions, *Ann. Probab.* **37**, 314 (2009).
- [33] P.J. Forrester, The limiting Kac random polynomial and truncated random orthogonal matrices, *J. Stat. Mech.* (2010) P12018.
- [34] B.A. Khoruzhenko, H.-J. Sommers, and K. Zyczkowski, Truncations of random orthogonal matrices, *Phys. Rev. E* **82**, 040106 (2010).
- [35] F. Calegari and Z. Huang, Counting Perron numbers by absolute value, *J. Lond. Math. Soc.* **96**, 181 (2017).
- [36] NIST Digital Library of Mathematical Functions, <https://dlmf.nist.gov/26.8#i>.
- [37] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.121.150601>, which also cites Refs. [38–41], for the details of the computations.
- [38] L.C.G. del Molino, K. Pakdaman, J. Touboul, and G. Wainrib, The Real Ginibre Ensemble with $k = O(n)$ Real Eigenvalues, *J. Stat. Phys.* **163**, 303 (2016).
- [39] P.J. Forrester and S. Kumar, The probability that all eigenvalues are real for products of truncated real orthogonal random matrices, *J. Theor. Probab.* **1** (2017).
- [40] J.M. Hammersley, The zeros of a random polynomial, in *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability* (University of California Press, California, 1972), Vol. 2, pp. 89–111.
- [41] M. Ishikawa, H. Kawamuko, and S. Okada, A Pfaffian-Hafnian analogue of Borchardts identity, *Electron. J. Comb.* **12**, 1 (2005).
- [42] K. Zyczkowski and H.-J. Sommers, Truncations of random unitary matrices, *J. Phys. A* **33**, 2045 (2000).
- [43] B. Lacroix-A-Chez-Toine, A. Grabsch, S.N. Majumdar, and G. Schehr, Extremes of 2d Coulomb gas: universal intermediate deviation regime, *J. Stat. Mech.* (2018) 013203.
- [44] J.B. Hough, M. Krishnapur, Y. Peres, and B. Viràg, *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*, University Lecture Series (American Mathematical Society, Providence, 2009), Vol. 51.
- [45] E. Kanzieper, M. Poplavskyi, C. Time, R. Tribe, and O. Zeitouni, What is the probability that a large random matrix has no real eigenvalues?, *Ann. Appl. Probab.* **26**, 2733 (2016).
- [46] A. Borodin and C.D. Sinclair, The ginibre ensemble of real random matrices and its scaling limits, *Commun. Math. Phys.* **291**, 177 (2009).
- [47] C.D. Sinclair, Averages over Ginibre's ensemble of random real matrices, *Int. Math. Res. Not.* **2007**, rnm015 (2007).
- [48] H.J. Sommers, Symplectic structure of the real Ginibre ensemble, *J. Phys. A* **40**, F671 (2007).
- [49] P.J. Forrester and T. Nagao, Skew orthogonal polynomials and the partly symmetric real Ginibre ensemble, *J. Phys. A* **41**, 375003 (2008).
- [50] G. Akemann and E. Kanzieper, Statistics of Real Eigenvalues in Ginibres Ensemble of Random Real Matrices, *Phys. Rev. Lett.* **95**, 230201 (2005).
- [51] G. Akemann and E. Kanzieper, Integrable structure of Ginibres ensemble of real random matrices and a Pfaffian integration theorem, *J. Stat. Phys.* **129**, 1159 (2007).
- [52] R. Tribe and O. Zeitouni, Pfaffian formulae for one dimensional coalescing and annihilating systems, *Electron. J. Probab.* **16**, 2080 (2011).
- [53] We recall that, for an antisymmetric matrix A of size $2m \times 2m$, the Pfaffian is defined as $\text{Pf}(A) = 1/(2^m m!) \sum_{\sigma \in S_{2m}} \epsilon(\sigma) \prod_{i=1}^m a_{\sigma(2i-1), \sigma(2i)}$, where S_{2m} is the group of permutations of $2m$ elements.
- [54] M. Gebert and M. Poplavskyi, On pure complex spectrum for truncations of random orthogonal matrices and Kac polynomials (to be published).
- [55] P.J. Forrester, J.R. Ipsen, and S. Kumar, How many eigenvalues of a product of truncated orthogonal matrices are real?, [arXiv:1708.00967](https://arxiv.org/abs/1708.00967) [Exper. Math. (to be published)].
- [56] H. Widom, Hankel matrices, *Trans. Am. Math. Soc.* **121**, 1 (1966).
- [57] Note that the computation of $\varphi(x)$ for $x > 1/\pi$ requires an analytical continuation of $\psi(s)$ for $s > 0$, which can be done straightforwardly from its explicit expression in Eq. (19).

- [58] The “clipped” process has obviously the same zero-crossings properties as $X(t)$ itself.
- [59] S. Matsumoto and T. Shirai, Correlation functions for zeros of a Gaussian power series and Pfaffians, [Electron. J. Pro](#) **18**, 1 (2013).
- [60] Note that σ_0 is thus a smooth process, which is at variance with σ_i with $i \gg 1$, which is nonsmooth.
- [61] T. Ohta, D. Jasnow, and K. Kawasaki, Universal Scaling in the Motion of Random Interfaces, [Phys. Rev. Lett.](#) **49**, 1223 (1982).
- [62] A. J. Bray, Theory of phase-ordering kinetics, [Adv. Phys.](#) **51**, 481 (2002).

Exact persistence exponent for the $2d$ -diffusion equation and related Kac polynomials:

Supplementary material

Mihail Poplavskyi¹ and Grégory Schehr²

¹*King's College London, Department of Mathematics, London WC2R 2LS, United Kingdom*

²*LPTMS, CNRS, Univ. Paris-Sud, Université Paris-Saclay, 91405 Orsay, France*

We give the principal details of the calculations described in the main text of the Letter.

1) DERIVATION OF EQS. (9) AND (10) IN THE MAIN TEXT

In this section we discuss the ideas behind the formulae (9) and (10) of the main text in order to give a clear picture of the connection between truncations of random orthogonal matrices and Kac polynomials. Let O be a $(2n+1) \times (2n+1)$ orthogonal matrix decomposed as [see Eq. (8) in the main text]

$$O = \begin{pmatrix} M_{2n} & \mathbf{u} \\ \mathbf{v}^T & a \end{pmatrix}, \quad (1)$$

where \mathbf{u} and \mathbf{v} are column vectors of length n and a is a scalar. We are interested in the eigenvalues of M_{2n} that can be found as the roots of the characteristic polynomial $g_M(z) = \det(z\mathbb{I}_{2n} - M_{2n})$.

We first recall that, for a block matrix U of the form

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (2)$$

where A, B, C and D are matrices (with A and D invertible), the determinant $\det U$ can be written in two different ways as

$$\det U = \det A \det(D - CA^{-1}B) \quad (3)$$

$$= \det D \det(A - BD^{-1}C). \quad (4)$$

Let us consider the following $(2n+1) \times (2n+1)$ rectangular matrix X [1]

$$X = \begin{pmatrix} \mathbb{I}_{2n} - zM_{2n} & \mathbf{u} \\ -\mathbf{v}^T & z^{-1}a \end{pmatrix},$$

then for $a \neq 0$ and assuming that $\mathbb{I}_{2n} - zM_{2n}$ is an invertible matrix, one can write (using Eqs. (3) and (4))

$$\det X = \det(\mathbb{I}_{2n} - zM_{2n}) \left(z^{-1}a + \mathbf{v}^T (\mathbb{I}_{2n} - zM_{2n})^{-1} \mathbf{u} \right) \quad (5)$$

$$= z^{-1}a \det(\mathbb{I}_{2n} - zM_{2n} + za^{-1}\mathbf{u}\mathbf{v}^T). \quad (6)$$

In Eq. (6) one recognizes the inverse of the top left corner of O^{-1} . Indeed, the block inversion formula gives

$$O^{-1} = \begin{pmatrix} (M_{2n} - a^{-1}\mathbf{u}\mathbf{v}^T)^{-1} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \quad (7)$$

where the matrices \tilde{B}, \tilde{C} and \tilde{D} are not needed here. Since O is an orthogonal matrix, $O^{-1} = O^T$ and hence (from Eqs. (1) and (7)) we obtain

$$(M_{2n} - a^{-1}\mathbf{u}\mathbf{v}^T)^{-1} = M_{2n}^T \implies M_{2n} - a^{-1}\mathbf{u}\mathbf{v}^T = (M_{2n}^{-1})^T \quad (8)$$

Therefore, Eq. (6) can be written as

$$\det X = z^{-1}a \det(\mathbb{I}_{2n} - z(M_{2n}^{-1})^T) = z^{-1}a \det M_{2n}^{-1} \det(M_{2n} - z\mathbb{I}_{2n}). \quad (9)$$

This yields

$$\frac{\det(z\mathbb{I}_{2n} - M_{2n})}{\det(zM_{2n} - \mathbb{I}_{2n})} = \det M_{2n} \left(1 + za^{-1}\mathbf{v}^T (\mathbb{I}_{2n} - zM_{2n})^{-1} \mathbf{u} \right) = \frac{\det M_{2n}}{a} \left(a + z\mathbf{v}^T (\mathbb{I}_{2n} - zM_{2n})^{-1} \mathbf{u} \right). \quad (10)$$

Notice that the inverse of the block matrix O in (1) can also be written as

$$O^{-1} = \begin{pmatrix} \tilde{A}' & \tilde{B}' \\ \tilde{C}' & (a - \mathbf{v}^T M_{2n}^{-1} \mathbf{u})^{-1} \end{pmatrix}, \quad (11)$$

where the matrices \tilde{A}', \tilde{B}' and \tilde{C}' are not needed here. Since O is an orthogonal matrix, $O^{-1} = O^T$ and hence (from Eqs. (1) and (11)) we obtain

$$a = (a - \mathbf{v}^T M_{2n}^{-1} \mathbf{u})^{-1}. \quad (12)$$

On the other hand, applying the formula in Eq. (3) to the block matrix O in (1), we obtain

$$\det O = \det M_{2n} \det(a - \mathbf{v}^T M_{2n}^{-1} \mathbf{u}) = \frac{\det M_{2n}}{a}, \quad (13)$$

where, in the last equality, we have used the identity (12). Finally, by combining Eq. (10) and (13), we obtain the formula given in Eq. (9) in the main text.

With probability one the eigenvalues of M_{2n} are interior points of the unit disc, since $M_{2n} M_{2n}^T = \mathbb{I} - \mathbf{u} \mathbf{u}^T < \mathbb{I}$. For $|z| < 1$ one can write the r.h.s. of Eq. (9) in the main text as a series (up to the sign of $\det O$)

$$F_{2n}(z) = a + \sum_{k=1}^{\infty} z^k \mathbf{v}^T M_{2n}^{k-1} \mathbf{u}. \quad (14)$$

All the roots of the series which are inside of the unit circle are the eigenvalues of matrix M_{2n} and vice versa. Finally, one can show [2] that the scaled sequence of the coefficients of the series F_{2n} given by

$$\sqrt{2n} \{a, \mathbf{v}^T \mathbf{u}, \mathbf{v}^T M_{2n} \mathbf{u}, \mathbf{v}^T M_{2n}^2 \mathbf{u}, \dots\}$$

converges, as $n \rightarrow \infty$, to the sequence of i.i.d. Gaussian random variables, with zero mean and unit variance. This implies that, for large N , the eigenvalues of truncated random orthogonal matrices behave identically to zeros of random Kac series, studied in [4] and discussed above. And one can check that, if we put $n \rightarrow \infty$ in the kernel of the Pfaffian point process describing the distribution of eigenvalues of random matrices M_{2n} , then the corresponding limiting kernel coincides with the one obtained in [4] (see, [3] for details).

Interestingly in a recent preprint [5], the authors, using similar ideas, found another models of random polynomials/series (related to the complex Kac polynomials/series) whose zeros behave as the eigenvalues of a corresponding ensemble of random matrices. This can be thought as another confirmation of a strong connection between the zeros of random functions and eigenvalues of random matrices.

2) DERIVATION OF THE FORMULA GIVEN IN EQ. (17) IN THE MAIN TEXT

We present the main steps leading to Eq. (17) in the main text and refer reader to [6] for further details. Let M_{2n} be a truncated random orthogonal matrix, as defined in Eq. (1). Its spectrum consists of ℓ real eigenvalues $\lambda_1 \leq \dots \leq \lambda_\ell$ and $m = n - \ell/2$ pairs of complex conjugate eigenvalues $z_{1,2} = x_1 \pm iy_1, \dots, z_{2m-1,2m} = x_m \pm iy_m$ whose joint distribution was derived in [7] and given by Eq. (12) in the text. It is well-known in random matrix theory (see, eg [8, 9]) that such conditional distributions give rise to Pfaffian point processes which satisfy Eq. (15) in the text [9]. Choosing the polynomials

$$P_{2j}(z) = z^{2j}, P_{2j+1}(z) = z^{2j+1} - \frac{2j}{2j+1} z^{2j-1}, \quad (15)$$

which are orthogonal with respect to the skew product in Eq. (16) in the main text [3], we get

$$\left\langle \prod_{\zeta \in \text{spec } M_{2n}} f(\zeta) \right\rangle_{M_{2n}} = \text{Pf}(U_f) / \text{Pf}(U_1),$$

where

$$\begin{aligned} (U_f)_{i,j} &= \int_{\mathbb{R}^2} \text{sgn}(y-x) f(x) f(y) P_{i-1}(x) P_{j-1}(y) w(x) w(y) dx dy \\ &\quad + 2i \int_{\mathbb{C}} f(z) f(\bar{z}) P_{i-1}(z) P_{j-1}(\bar{z}) \text{sgn}(\Im z) w(z) w(\bar{z}) d^2 z, \\ (U_1)_{i,j} &= \int_{\mathbb{R}^2} \text{sgn}(y-x) P_{i-1}(x) P_{j-1}(y) w(x) w(y) dx dy + 2i \int_{\mathbb{C}} P_{i-1}(z) P_{j-1}(\bar{z}) \text{sgn}(\Im z) w(z) w(\bar{z}) d^2 z. \end{aligned}$$

Let us denote by $\left(U_1^{(r)}\right)_{i,j}$, $\left(U_1^{(c)}\right)_{i,j}$ the two integrals with respect to real and complex variables in the right hand side (r.h.s.) of the above expressions. Then one can see that with $f(z) = 1 - (1 - e^s)\chi_{\mathbb{R}}(z)$ we have

$$(U_f)_{i,j} = e^{2s} \left(U_1^{(r)}\right)_{i,j} + \left(U_1^{(c)}\right)_{i,j} = (U_1)_{i,j} + (e^{2s} - 1) \left(U_1^{(r)}\right)_{i,j}. \quad (16)$$

We now focus on the computation of both $\left(U^{(r)}\right)_{i,j}$ and $\left(U^{(c)}\right)_{i,j}$ for $i, j = 1, 2, \dots, 2n$. First we note that both integrals vanish if i and j are of the same parity. This can be seen from

$$\begin{aligned} \left(U^{(r)}\right)_{i,j} &= \frac{1}{2\pi} \int_{-1}^1 \int_{-1}^1 \operatorname{sgn}(y-x) P_{i-1}(x) P_{j-1}(y) (1-x^2)^{-\frac{1}{2}} (1-y^2)^{-\frac{1}{2}} dx dy \\ \left| x, y \rightarrow -x, -y \right| &= \frac{1}{2\pi} (-1)^{i+j} \int_{-1}^1 \int_{-1}^1 \operatorname{sgn}(x-y) P_{i-1}(x) P_{j-1}(y) (1-x^2)^{-\frac{1}{2}} (1-y^2)^{-\frac{1}{2}} dx dy = - \left(U^{(r)}\right)_{i,j}. \end{aligned}$$

For complex part we write

$$\left(U^{(c)}\right)_{i,j} = \frac{i}{\pi} \iint_{\mathbb{D}} \operatorname{sgn}(y) P_{i-1}(x+iy) P_{j-1}(x-iy) \frac{dx dy}{((1-x^2+y^2)^2 + 4x^2 y^2)^{1/2}}.$$

After expanding both polynomials in powers of x, y we can see that the terms with odd powers of x and even powers of y cancel due to the symmetry of integrand. If i and j have the same parity, then there are no other terms in the expansion. Besides, it is easy to see that both $\left(U^{(r,c)}\right)_{i,j}$ are skew symmetric by interchanging $i \leftrightarrow j$ and we study below only the terms of the form $\left(U^{(r,c)}\right)_{2p+1, 2q+2}$. For the real part we obtain

$$\begin{aligned} \left(U^{(r)}\right)_{2p+1, 2q+2} &= \frac{1}{2\pi} \int_{-1}^1 \int_{-1}^1 \operatorname{sgn}(y-x) x^{2p} \left(y^{2q+1} - \frac{2q}{2q+1} y^{2q-1} \right) \frac{dx dy}{\sqrt{1-x^2} \sqrt{1-y^2}} \\ &= \frac{1}{2\pi(2q+1)} \int_{-1}^1 x^{2p} \frac{dx}{\sqrt{1-x^2}} \int_{-1}^1 \operatorname{sgn}(y-x) \frac{d}{dy} \left(-y^{2q} \sqrt{1-y^2} \right) dy \\ &= \frac{1}{\pi(2q+1)} \int_{-1}^1 x^{2p+2q} dx = \frac{2}{\pi(2q+1)(2p+2q+1)}. \end{aligned} \quad (17)$$

For the complex part we write

$$\left(U^{(c)}\right)_{2p+1, 2q+2} = \frac{2i}{\pi} \iint_{\mathbb{D}_+} (x+iy)^{2p} \left((x-iy)^{2q+1} - \frac{2q}{2q+1} (x-iy)^{2q-1} \right) \frac{dx dy}{((1-x^2+y^2)^2 + 4x^2 y^2)^{1/2}}, \quad (18)$$

where \mathbb{D}_+ is an upper half unit disk, and use

$$\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (x-iy)^{2q} \frac{\sqrt{(1-x^2+y^2)^2 + 4x^2 y^2}}{1-(x+iy)^2} = -2(2q+1) \frac{\left((x-iy)^{2q+1} - \frac{2q}{2q+1} (x-iy)^{2q-1} \right)}{((1-x^2+y^2)^2 + 4x^2 y^2)^{1/2}}.$$

Performing the integration in (18) we get

$$\begin{aligned}
\left(U^{(c)}\right)_{2p+1,2q+2} &= -\frac{i}{\pi(2q+1)} \left[\int_0^1 dy (x+iy)^{2p} (x-iy)^{2q} \frac{\sqrt{(1-x^2+y^2)^2 + 4x^2y^2}}{1-(x+iy)^2} \Big|_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} \right. \\
&\quad \left. + i \int_{-1}^1 dx (x+iy)^{2p} (x-iy)^{2q} \frac{\sqrt{(1-x^2+y^2)^2 + 4x^2y^2}}{1-(x+iy)^2} \Big|_{y=0}^{y=\sqrt{1-x^2}} \right] \\
&= \frac{1}{\pi(2q+1)} (1+(-1)^{p-q}) \int_{-1}^1 \left(\sqrt{1-x^2} + ix \right)^{2p-1} \left(\sqrt{1-x^2} - ix \right)^{2q} dx - \frac{1}{\pi(2q+1)} \int_{-1}^1 x^{2p+2q} dx \\
&= \frac{1}{(2q+1)} \delta_{p,q} - \frac{2}{\pi(2q+1)(2p+2q+1)} .
\end{aligned} \tag{19}$$

By adding the real (17) and complex (19) parts together we obtain [using Eq. (16)] $(U_1)_{i,j} = \frac{1}{j-1}$, if $i = j-1$ is odd, $(U_1)_{i,j} = -\frac{1}{j}$, if $i = j+1$ is even and zero otherwise. One can then easily check that

$$\text{Pf } U_1 = \prod_{j=1}^n (2j-1)^{-1},$$

and use the above formulae in Eqs. (17) and (19), together with Eq. (16), to show the formula (17) given in the text.

3) DERIVATION OF THE FORMULA FOR THE CUMULANTS GIVEN IN EQ. (7) OF THE MAIN TEXT

In this section, we derive the expression for the cumulants κ_p of the number of zero crossings N_t of the $2d$ -diffusing field up to time t , given in Eq. (7) in the main text. As explained in the text, the statistical properties of N_t for large t are obtained from the number of positive real eigenvalues \mathcal{N}_n^+ of the truncated orthogonal random matrix M_{2n} [see Eq. (1)] for which we have shown in the paper that

$$\langle e^{s\mathcal{N}_n^+} \rangle_{M_{2n}} \sim n^{\frac{\psi(s)}{2}} \tag{20}$$

where $\psi(s)$ is given in Eq. (19) in the main text. The cumulants of \mathcal{N}_n^+ , denoted as $\langle [\mathcal{N}_n^+]^p \rangle_{M_{2n}}^c$ can be obtained from the following expansion

$$\ln \langle e^{s\mathcal{N}_n^+} \rangle_{M_{2n}} = \sum_{p=1}^{\infty} \frac{s^p}{p!} \langle [\mathcal{N}_n^+]^p \rangle_{M_{2n}}^c \sim \frac{\psi(s)}{2} \ln n, \tag{21}$$

where in the second estimation we have used (20). Therefore, to compute these cumulants, we need to expand $\psi(s)$ for small s . It is then convenient to start from Eq. (18) and expand the logarithm to get

$$\psi(s) = -\frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{1}{m} (1-e^{2s})^m \int_0^{\infty} du \operatorname{sech}^m \left(\frac{\pi u}{2} \right) = -\frac{1}{2\pi^2} \sum_{m=1}^{\infty} \frac{1}{m} B \left(\frac{m}{2}, \frac{1}{2} \right) (1-e^{2s})^m, \tag{22}$$

valid for $s \leq 0$ where we have used

$$\int_0^{\infty} du \operatorname{sech}^m \left(\frac{\pi u}{2} \right) = \frac{1}{\pi} B \left(\frac{m}{2}, \frac{1}{2} \right), \tag{23}$$

with $B(a,b)$ being the Euler beta function. We now expand $(e^{2s}-1)^m$ in (22), which yields

$$(e^{2s}-1)^m = \left(\sum_{p=1}^{\infty} \frac{(2s)^p}{p!} \right)^m = \sum_{p=m}^{\infty} \frac{(2s)^p}{p!} \sum_{p_1+\dots+p_m=p, p_i \geq 1} \frac{p!}{p_1! p_2! \dots p_m!}. \tag{24}$$

Let us denote the last sum by $c_{m,p}$. We first notice that the binomial theorem yields

$$\sum_{p_1+\dots+p_m=p, p_i \geq 0} \frac{p!}{p_1!p_2!\dots p_m!} = (1+1+\dots+1)^p = m^p.$$

However, to calculate $c_{m,p}$ we need to subtract all terms with $p_i = 0$ for at least one $1 \leq i \leq m$. For this purpose, we use the inclusion-exclusion principle as follows. First we subtract all the terms with $p_j = 0$ for a given value of j . There are m choices of index j and all terms with $p_j = 0$ sum up to

$$\sum_{\substack{p_1+\dots+p_m=p \\ p_j=0, \forall i \neq j \\ p_i \geq 0}} \frac{p!}{p_1!p_2!\dots p_m!} = (m-1)^p.$$

Next we add all the terms containing $p_j = p_k = 0$ for a given pair $j < k$. These terms sum up to

$$\sum_{\substack{p_1+\dots+p_m=p \\ p_j=p_k=0, \forall i \neq j,k \\ p_i \geq 0}} \frac{p!}{p_1!p_2!\dots p_m!} = (m-2)^p,$$

and there are $\binom{m}{2}$ choices of pair of indices $j < k$. Continuing the calculation we obtain

$$c_{m,p} = m^p - m(m-1)^p + \binom{m}{2}(m-2)^p - \dots = \sum_{\ell=0}^m (-1)^\ell \binom{m}{\ell} (m-\ell)^p = m! \mathcal{S}_p^{(m)}, \quad (25)$$

where, in the last equality, we have used the definition of Stirling number of the second kind $\mathcal{S}_p^{(m)}$ [10]. Combining Eqs. (22), (24) and (25) we find

$$\psi(s) = -\frac{1}{2\pi^2} \sum_{m=1}^{\infty} (-1)^m (m-1)! B\left(\frac{m}{2}, \frac{1}{2}\right) \sum_{p=m}^{\infty} \frac{(2s)^p}{p!} \mathcal{S}_p^{(m)}. \quad (26)$$

One can use induction and the recurrence relations for Stirling numbers to show

$$\mathcal{S}_p^{(m)} \leq 2^p \frac{p!}{m!}, \quad (27)$$

which, together with the asymptotic behaviour for $B\left(\frac{m}{2}, \frac{1}{2}\right) \sim \sqrt{\frac{2\pi}{m}}$ for $m \gg 1$, yields that the double infinite sum in (26) is absolutely convergent for small values of s . One can thus interchange the order of the summation over m and p to obtain

$$\psi(s) = \sum_{p=1}^{\infty} s^p \frac{2^{p-1}}{\pi^2 p!} \sum_{m=1}^p (-2)^{m-1} \mathcal{S}_p^{(m)} \Gamma^2\left(\frac{m}{2}\right). \quad (28)$$

Finally, by combining Eqs. (21) together with the small s expansion of $\psi(s)$ in (28), we obtain that $\langle [\mathcal{N}_n^+]^p \rangle_{M_{2n}}^c \sim \kappa_p \ln n$ where the expression for the cumulants κ_p is given in Eq. (7) in the main text.

4) ASYMPTOTIC EXPANSION OF THE LARGE DEVIATION FUNCTION $\varphi(x)$

In this section, we study the large deviation function $\varphi(x)$ defined in Eq. (5) in the main text and obtain its asymptotic behaviours given in (6) in the main text. This function $\varphi(x)$ is defined in term from $\psi(s)$ given in Eq. (19) of the text

$$\varphi(x) = \max_{s \in \mathbb{R}} \left(s x - \frac{\psi(s)}{2} \right), \quad \psi(s) = \frac{1}{8} - \frac{2}{\pi^2} \left(\cos^{-1} \left(\frac{e^s}{\sqrt{2}} \right) \right)^2. \quad (29)$$

Note that the function $\psi(s)$, initially defined for $s < 0$ (see Eq. (18) in the text) can be analytically continued to any positive $s > 0$, where $\psi(s)$ is actually also real. The maximum of $(sx - \frac{\psi(s)}{2})$ in Eq. (29) is attained at a unique point s^* which satisfies

$$x - \frac{1}{2}\psi'(s^*) = 0 \iff x = \mathcal{F}(z^*) \text{ with } \begin{cases} z^* = \frac{e^{s^*}}{\sqrt{2}} \\ \mathcal{F}(z) = \frac{2z}{\pi^2\sqrt{1-z^2}} \cos^{-1}(z) . \end{cases} \quad (30)$$

Note that the function $\mathcal{F}(z)$ is a monotonously increasing function such that $\mathcal{F}(z \rightarrow 0) \rightarrow 0$ and $\mathcal{F}(z \rightarrow \infty) \rightarrow \infty$ such that its inverse $\mathcal{F}^{-1}(z)$ is well defined for any $z \in \mathbb{R}^+$. In particular, one can show that its asymptotic behaviours are given by

$$\mathcal{F}^{-1}(z) \sim \begin{cases} \pi z + 2\pi z^2, & z \rightarrow 0 \\ \frac{1}{2} e^{\frac{\pi^2}{2}} z, & z \rightarrow \infty . \end{cases} \quad (31)$$

Therefore using Eqs. (29) and (30), we obtain that $\varphi(x)$ reads

$$\varphi(x) = s^*x - \frac{\psi(s^*)}{2} \quad (32)$$

$$= x \ln(\sqrt{2}z^*) - \frac{1}{16} - \frac{\pi^2}{4}x^2 + \frac{\pi^2}{4}\left(\frac{x}{z^*}\right)^2, \quad z^* = \mathcal{F}^{-1}(x), \quad (33)$$

where, in (33), we have used the equation (30). Using this expression (33) together with the asymptotic behaviours of $\mathcal{F}^{-1}(x)$ in Eq. (31), we obtain the asymptotic behaviours of $\varphi(x)$ as

$$\varphi(x) \sim \begin{cases} \frac{3}{16} + x \ln x + x(\ln(\pi\sqrt{2}) - 1), & x \rightarrow 0 \\ \frac{\pi^2}{4}x^2 - \frac{x}{2} \ln 2 - \frac{1}{16}, & x \rightarrow \infty \end{cases} \quad (34)$$

the first terms of which give the behaviours given in Eq. (6) in the text.

We complete our analysis of the large deviation function $\varphi(x)$ by analysing its behaviour near its minimum, which is reached for $x = 1/(2\pi)$ – this thus corresponds to the peak of the distribution of the number of zeros. For $x = 1/(2\pi)$, it is easy to show that the maximum of $sx - \psi(s)/2$ is reached for $s^* = 0$, which, together with Eq. (29) shows that $\varphi(x = 1/(2\pi)) = 0$. Furthermore, for x close to $1/(2\pi)$, i.e. for $0 < |x - 1/(2\pi)| \ll 1$, one can show, by solving Eq. (30) perturbatively, that s^* behaves as

$$s^* = \frac{\pi^2}{\pi - 2} \left(x - \frac{1}{2\pi} \right) + \mathcal{O} \left(\left(x - \frac{1}{2\pi} \right)^2 \right). \quad (35)$$

By injecting this behavior (35) into Eq. (29), i.e. $\varphi(x) = s^*x - \psi(s^*)/2$, one finds

$$\varphi(x) = \frac{\pi^2}{2(\pi - 2)} \left(x - \frac{1}{2\pi} \right)^2 + \mathcal{O} \left(\left(x - \frac{1}{2\pi} \right)^3 \right), \quad (36)$$

as announced in the second line of Eq. (6) in the text.

We end this section by a remark: here we have studied the distribution $\tilde{p}_k(n)$, with $k = 0, \dots, 2n$, of the number \mathcal{N}_n^+ of the real (positive) eigenvalues of M_{2n} in the limit where both k and n are large, and $k = O(\ln n)$, and this regime $\tilde{p}_k(n) \sim n^{-\varphi(k/\ln n)}$. However, there might well exist different scaling regimes, for instance for $k = O(n)$, as in the case of the real Ginibre matrices [11]. In fact, for truncated orthogonal matrices the case $k = 2n$ was investigated in [12] and it would be interesting to study the generic case $\alpha = k/(2n)$, for k and n large.

5) REAL RANDOM SERIES AS A PFAFFIAN POINT PROCESS AND THE FORMULAE (20)-(21) OF THE MAIN TEXT

In this section, we recall the main results obtained in Ref. [4] and derive the expression given in Eqs. (20) and (21) in the text. In Ref. [4] where the authors considered the "limiting case" of Kac polynomials of degree $n \rightarrow \infty$ and

studied the distribution of the associated roots. More precisely, let $\{a_k\}_{k=0}^{\infty}$ be a sequence of i.i.d. real standard Gaussian random variables and let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

be an analytic function inside the unit disk \mathbb{D} . As a function of the real parameter $x \in]-1, 1[$, $\{f(x)\}_{-1 < x < 1}$ is a real Gaussian process with covariance kernel $R(x, y) = \mathbb{E}[f(x)f(y)] = (1 - xy)^{-1}$. The real zeros of f are random points on $(-1, 1)$ and their distribution can be studied by using the remarkable formula due to Hammersley [13] for the m -point correlation function $\rho_m(x_1, \dots, x_m)$ of real zeros. It reads

$$\rho_m(x_1, x_2, \dots, x_m) = \frac{\mathbb{E}[|f'(x_1)f'(x_2) \dots f'(x_m)| |f(x_1) = f(x_2) = \dots = f(x_m) = 0]}{(2\pi)^{m/2} \sqrt{\det(R(x_i, x_j))}}, \quad (37)$$

The r.h.s. can be also rewritten by noting that

$$\begin{aligned} & \left(\frac{2}{\pi}\right)^{m/2} \frac{1}{\sqrt{\det R(x_i, x_j)}} \mathbb{E}[|f'(x_1)f'(x_2) \dots f'(x_m)| |f(x_1) = f(x_2) = \dots = f(x_m) = 0] \\ &= \lim_{s_1 \rightarrow x_1 + 0} \lim_{s_2 \rightarrow x_2 + 0} \dots \lim_{s_m \rightarrow x_m + 0} \frac{\partial^m}{\partial x_1 \partial x_2 \dots \partial x_m} \mathbb{E}[\text{sgn } f(x_1) \text{sgn } f(x_2) \dots \text{sgn } f(x_m) \text{sgn } f(s_1) \text{sgn } f(s_2) \dots \text{sgn } f(s_m)]. \end{aligned} \quad (38)$$

The above formula clearly shows the importance of the so-called sign-correlation function for the Gaussian process $f(t)$ defined as

$$S(x_1, x_2, \dots, x_{2k}) = \mathbb{E}[\text{sgn } f(x_1) \text{sgn } f(x_2) \dots \text{sgn } f(x_{2k})]. \quad (39)$$

The computation of the r.h.s. of (37) can then be carried out as follows. Since $f(\cdot)$ is a Gaussian process of covariance $R(x, y)$, one can show that the process $f(\cdot)$ conditioned on $f(x_0) = 0$ for some $x_0 \in]-1, 1[$, denoted as $(f(\cdot)|f(x_0) = 0)$ is also a Gaussian process. Let $x_1, x_2, \dots, x_m \in]-1, 1[$, then the Gaussianity of $f(\cdot)$ yields that the vector $\vec{f}_0 = (f(x_0), f(x_1), \dots, f(x_m))$ is normally distributed with zero mean and covariance matrix $\Sigma_0 = \{R(x_i, x_j)\}_{i,j=0}^m$, i.e. $p(\vec{f}_0) \propto \exp\left\{-\frac{1}{2}\vec{f}_0 \Sigma_0 \vec{f}_0^T\right\}$. A simple calculation yields the distribution of the vector $\vec{f} = (f(x_1), \dots, f(x_m))$ conditioned on the event $f(x_0) = 0$ as

$$p(\vec{f}) \propto \exp\left\{-\frac{1}{2}\vec{f} \Sigma_{22} \vec{f}^T\right\}, \quad \text{where} \quad \Sigma_0^{-1} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{pmatrix},$$

where Σ_{11} is a scalar, Σ_{12} is a row vector of size m and Σ_{22} is an $m \times m$ matrix. The block inversion formula yields

$$(\Sigma_{22})_{ij} = R(x_i, x_j) - R(x_i, x_0)R^{-1}(x_0, x_0)R(x_0, x_j).$$

This proves that the conditional process $(f(\cdot)|f(x_0) = 0)$ is Gaussian with covariance

$$\tilde{R}(x, y) = R(x, y) - \frac{R(x, x_0)R(x_0, y)}{R(x_0, x_0)}. \quad (40)$$

For the special case considered here where $R(x, y) = 1/(1 - xy)$, it is easy to check that

$$\tilde{R}(x, y) = R(x, y) - \frac{R(x, x_0)R(y, x_0)}{R(x_0, x_0)} = R(x, y)\mu(x, x_0)\mu(y, x_0), \quad \text{with} \quad \mu(x, x_0) = \frac{x - x_0}{1 - xx_0}, \quad (41)$$

which shows that $(f(\cdot)|f(x_0) = 0)$ is a Gaussian process equal in distribution to $\mu(\cdot, x_0)f(\cdot)$. This is a crucial property, satisfied by this special correlator $R(x, y) = 1/(1 - xy)$. Using the linearity of the derivative, one gets [4]

$$(f'(x_1), f'(x_2), \dots, f'(x_n)|f(x_1) = f(x_2) = \dots = f(x_m) = 0) \stackrel{d}{=} (M(x_1, \mathbf{x})f(x_1), M(x_2, \mathbf{x})f(x_2), \dots, M(x_m, \mathbf{x})f(x_m)), \quad (42)$$

where $M(\cdot, \mathbf{x}) = \left(\prod_i \mu(\cdot, x_i) \right)'$ with the derivative taken with respect to the first argument and hence, one can check that $M(x_i, \mathbf{x}) = \frac{1}{1-x_i^2} \prod_{j \neq i} \frac{x_i - x_j}{(1-x_i x_j)^2}$. Combining Eqs. (38) and (42), one gets, after some manipulations [4]

$$\begin{aligned} \frac{\partial^{2n}}{\partial x_1 \partial x_2 \dots \partial x_{2n}} S(x_1, x_2, \dots, x_{2n}) &= \left(\frac{2}{\pi} \right)^n \frac{1}{\sqrt{\det R(x_i, x_j)}} \prod_i M(x_i, \mathbf{t}) \mathbb{E}[f(x_1) f(x_2) \dots f(x_{2n})] \\ &= \left(\frac{2}{\pi} \right)^n \prod_{i < j} \text{sgn}(x_j - x_i) \text{Pf}(\mathbb{K}_{11}(x_i, x_j)) = \left(\frac{2}{\pi} \right)^n \prod_{i < j} \text{sgn}(x_j - x_i) \frac{\partial^{2n}}{\partial x_1 \partial x_2 \dots \partial x_{2n}} \text{Pf}(\mathbb{K}_{22}(x_i, x_j)), \end{aligned} \quad (43)$$

where

$$\mathbb{K}(x, y) = \begin{pmatrix} \mathbb{K}_{11}(x, y) & \mathbb{K}_{12}(x, y) \\ \mathbb{K}_{12}(x, y) & \mathbb{K}_{22}(x, y) \end{pmatrix} = \begin{pmatrix} \frac{x-y}{\sqrt{1-x^2}\sqrt{1-y^2}(1-xy)^2} & \sqrt{\frac{1-y^2}{1-x^2}} \frac{1}{1-xy} \\ -\sqrt{\frac{1-x^2}{1-y^2}} \frac{1}{1-xy} & \text{sgn}(x-y) \arcsin \frac{\sqrt{1-x^2}\sqrt{1-y^2}}{1-xy} \end{pmatrix}, \quad (44)$$

with $\mathbb{K}_{11}(x, y) = \frac{\partial^2}{\partial x \partial y} \mathbb{K}_{22}(x, y)$, $\mathbb{K}_{12}(x, y) = \frac{\partial}{\partial x} \mathbb{K}_{22}(x, y)$ and $\mathbb{K}_{21}(x, y) = \frac{\partial}{\partial y} \mathbb{K}_{22}(x, y)$. We refer the reader to Ref. [4] for the details of the computations leading to the Pfaffian expression given in Eqs. (43) and (44). Finally, using appropriate boundary values of both sign-correlation function and corresponding Pfaffian we can integrate over the x_i 's on both sides of (43) to get that, for any ordered $2n$ -tuples $-1 < x_1 < x_2 < \dots < x_{2n} < 1$, one has

$$S(x_1, x_2, \dots, x_{2n}) = \left(\frac{2}{\pi} \right)^n \text{Pf}(\mathbb{K}_{22}(x_i, x_j)). \quad (45)$$

We now use this identity (45) to compute $\rho_m(x_1, \dots, x_m)$ from Eqs. (37) and (38). It is then convenient to relabel the variables and use Eq. (45) with the relabelling $x_{2i+1} \rightarrow x_i$ and $x_{2i} = s_i$. With this relabelling, we can now differentiate Eq. (45) with respect to the variables of odd indices and then take the limit $x_{2i} \rightarrow x_{2i-1}$ for all $i = 1, \dots, m$, according to Eq. (38). This yields finally [4]

$$\rho_m(x_1, \dots, x_m) = \pi^{-m} \text{Pf}(\mathbb{K}(x_i, x_j)). \quad (46)$$

We can now use this result (46) to compute the multi-time sign-correlation function for the $2d$ -diffusion field. Indeed, the $2d$ -diffusion field $\varphi(\mathbf{0}, t)$ coincides, for large time $t \gg 1$, with the Kac polynomials $f(x)$ for x close 1 [14, 15]. More precisely, if one considers the normalised process $X(t) = \varphi(\mathbf{0}, t) / \langle \varphi^2(\mathbf{0}, t) \rangle$, one has

$$X(t) = \frac{\varphi(\mathbf{0}, t)}{\langle \varphi^2(\mathbf{0}, t) \rangle} \stackrel{d}{\simeq} f(x = 1 - 1/t), \quad t \rightarrow \infty. \quad (47)$$

Therefore, from Eqs. (39) and (45), one obtains that the multi-time correlation functions of $\text{sgn } X(t)$ also have a Pfaffian structure with kernel $\mathbb{K}_{22}(x_i = 1 - 1/t_i, x_j = 1 - 1/t_j)$ with $t_i, t_j \gg 1$. Finally, using that

$$\mathbb{K}_{22}(x_i = 1 - 1/t_i, x_j = 1 - 1/t_j) \approx \text{sgn}(t_j - t_i) \arcsin \left(\frac{2\sqrt{t_i t_j}}{t_i + t_j} \right), \quad t_i, t_j \gg 1, \quad (48)$$

we obtain the formulae given Eqs. (20) and (21) in the text. In particular, specifying Eqs. (45) and (48) to $m = 1$, we obtain

$$\langle X(t_1) X(t_2) \rangle \approx \text{sgn}(t_2 - t_1) \frac{2}{\pi} \arcsin \left(\frac{2\sqrt{t_1 t_2}}{t_1 + t_2} \right), \quad t_1, t_2 \gg 1. \quad (49)$$

6) PFAFFIAN STRUCTURE OF THE GSP WITH CORRELATOR $c(T) = \text{sech}(T/2)$

In this section we extend the ideas of [4] to another process that plays a crucial role in our study (see Fig. 1 in the main text), namely the stationary GSP $Y(T)$ with correlation function $c(T) = \text{sech}(T/2)$ and study the corresponding distribution of zeros. Similarly to (37) we compute the correlation functions of the zeros by studying the conditional process $(Y(\cdot) | Y(t) = 0)$, which is also a GSP with correlation function (see Eq. (40) in the previous section)

$$c(x - y) - \frac{c(x - t)c(y - t)}{c(0)} = \text{sech} \left(\frac{x - y}{2} \right) - \text{sech} \left(\frac{x - t}{2} \right) \text{sech} \left(\frac{y - t}{2} \right) = c(x - y) \tanh(x - t) \tanh(y - t),$$

where the last equality can be checked by using standard (hyperbolic) trigonometric relations. Therefore $(X(\cdot)|X(t) = 0) \stackrel{d}{=} \mu(\cdot, t)X(t)$ with $\mu(x, t) = \tanh \frac{x-t}{2}$. Now using the linearity of the derivative we obtain (cf. (42))

$$(f'(t_1), f'(t_2), \dots, f'(t_m) | f(t_1) = f(t_2) = \dots = f(t_m) = 0) \stackrel{d}{=} (M(t_1, \mathbf{t})f(t_1), M(t_2, \mathbf{t})f(t_2), \dots, M(t_m, \mathbf{t})f(t_m)),$$

with $M(\cdot, \mathbf{t}) = \left(\prod_i \mu(\cdot, t_i) \right)'$ and $M(t_i, \mathbf{t}) = \frac{1}{2} \prod_{j \neq i} \tanh \frac{t_i - t_j}{2}$. Using that $\frac{d}{dt} \operatorname{sgn}(t) = 2\delta(t)$, we obtain (for more details see [4] and references therein)

$$\begin{aligned} \frac{\partial^{2m}}{\partial t_1 \partial t_2 \dots \partial t_{2m}} \mathbb{E} [\operatorname{sgn} X(t_1) \dots \operatorname{sgn} X(t_{2m})] &= 2^{2m} \mathbb{E} [\delta(X(t_1)) X'(t_1) \dots \delta(X(t_{2m})) X'(t_{2m})] \\ &= \frac{2^m}{\pi^m \sqrt{\det c(t_i - t_j)}} \mathbb{E} [X'(t_1) \dots X'(t_{2m}) | X(t_1) = \dots = X(t_{2m}) = 0], \end{aligned}$$

where, in the last equality, we used the explicit expression for the probability density of vector the $(X(t_i))$ at 0. Using the previous result on the conditional process $(X(\cdot)|X(t) = 0)$ and the Wick theorem, we obtain (cf. (43))

$$\begin{aligned} \frac{\partial^{2m}}{\partial t_1 \partial t_2 \dots \partial t_{2m}} \mathbb{E} [\operatorname{sgn} X(t_1) \operatorname{sgn} X(t_2) \dots \operatorname{sgn} X(t_{2m})] &= \left(\frac{2}{\pi} \right)^m \frac{1}{\sqrt{\det \operatorname{sech} \frac{t_i - t_j}{2}}} \prod_i M(t_i, \mathbf{t}) \mathbb{E} [X(t_1) X(t_2) \dots X(t_{2m})] \\ &= (-2\pi)^{-m} \frac{\prod_{i < j} \tanh^2 \frac{t_i - t_j}{2}}{\sqrt{\det \operatorname{sech} \frac{t_i - t_j}{2}}} \operatorname{Hf} \left(\operatorname{sech} \frac{t_i - t_j}{2} \right), \end{aligned} \quad (50)$$

where $\operatorname{Hf}(A)$ – the so called Hafnian – for a $2m \times 2m$ symmetric matrix $A = (a_{i,j})_{1 \leq i,j \leq m}$ is defined as $\operatorname{Hf}(A) = 1/(2^m m!) \sum_{\sigma \in \mathcal{S}_{2m}} \prod_{i=1}^m a_{\sigma(2i-1), \sigma(2i)}$, where \mathcal{S}_{2m} is the group of permutations of $2m$ elements. The denominator in the above formula can be evaluated by using the following formula for a Cauchy's determinant

$$\det \left(\frac{1}{1 - x_i y_j} \right)_{i,j=1}^k = \frac{\prod_{1 \leq i < j \leq k} (x_i - x_j)(y_i - y_j)}{\prod_{1 \leq i, j \leq k} (1 - x_i y_j)},$$

with $x_i = e^{t_i}$ and $y_j = -e^{-t_j}$. We then obtain

$$\det \operatorname{sech} \frac{t_i - t_j}{2} = 2^{2m} \det \frac{1}{1 + e^{t_i - t_j}} = 2^{2m} \frac{\prod_{1 \leq i < j \leq 2m} 4 \sinh^2 \frac{t_i - t_j}{2}}{2^{2m} \prod_{1 \leq i < j \leq 2m} 4 \cosh^2 \frac{t_i - t_j}{2}} = \prod_{1 \leq i < j \leq 2m} \tanh^2 \frac{t_i - t_j}{2},$$

and after combining with Eq. (50), we get

$$\frac{\partial^{2m}}{\partial t_1 \partial t_2 \dots \partial t_{2m}} \mathbb{E} [\operatorname{sgn} X(t_1) \operatorname{sgn} X(t_2) \dots \operatorname{sgn} X(t_{2m})] = (2\pi)^{-m} \prod_{1 \leq i < j \leq 2m} \operatorname{sgn}(t_j - t_i) \tanh \frac{t_i - t_j}{2} \operatorname{Hf} \left(\operatorname{sech} \frac{t_i - t_j}{2} \right).$$

Using a formula due to Ishikawa, Kawamuko, and Okada [16]

$$\prod_{1 \leq i < j \leq 2n} \frac{x_i - x_j}{x_i + x_j} \operatorname{Hf} \left(\frac{1}{x_i + x_j} \right)_{i,j=1}^{2n} = \operatorname{Pf} \left(\frac{x_i - x_j}{(x_i + x_j)^2} \right)_{i,j=1}^{2n}, \quad (51)$$

we obtain the derivative of the multi-time spin-correlation function under as a Pfaffian. Indeed, by taking $x_i = e^{t_i}$ we obtain

$$\frac{\partial^{2m}}{\partial t_1 \partial t_2 \dots \partial t_{2m}} \mathbb{E} [\operatorname{sgn} X(t_1) \dots \operatorname{sgn} X(t_{2m})] = (2\pi)^{-m} \prod_{i < j} \operatorname{sgn}(t_j - t_i) \operatorname{Pf} \left(\frac{\sinh \frac{t_i - t_j}{2}}{\cosh^2 \frac{t_i - t_j}{2}} \right).$$

Spin correlation function can now be obtained by direct integration. As a boundary condition one should use decay of correlation functions at infinity. All together this implies that for $t_1 < t_2 < \dots < t_{2m}$

$$\mathbb{E} [\operatorname{sgn} X(t_1) \operatorname{sgn} X(t_2) \dots \operatorname{sgn} X(t_{2m})] = (2\pi)^{-m} \operatorname{Pf} \left(8 \arctan \tanh \frac{t_i - t_j}{4} - 2\pi \operatorname{sgn}(t_i - t_j) \right).$$

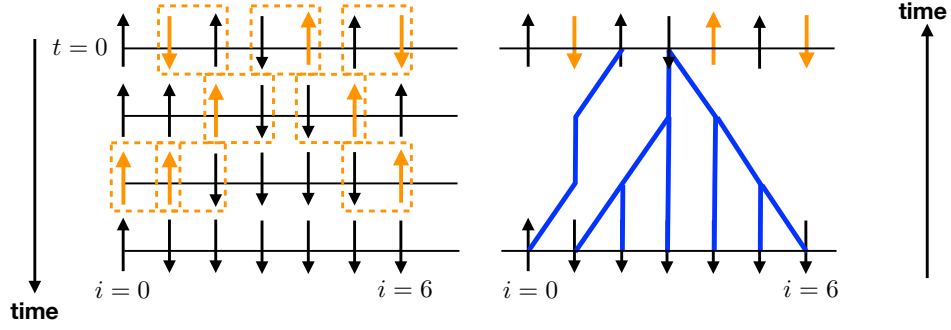


FIG. 1. Illustration of the mapping between the Glauber dynamics (53), in the left panel, and coalescing paths, in the right panel. **Left panel:** at each time steps, the state of the orange spins are changed (while the black spins stay the same) and they take the value of the neighbouring spin inside the square box. **Right panel:** the arrow of time is reversed such that the initial configuration of the walks is actually the final state of the spin system. At each time step a given path connects the sites i and j (with $j = i - 1, i$ or $i + 1$) such that σ_j is the “ancestor” of σ_i . For instance, during the final step in the left panel, the spin at site $i = 0$ took the value of the neighbouring spin at site $i = 1$, hence the first step of the walk that starts at $i = 0$ goes from $i = 0$ to $i = 1$.

Performing differentiation and taking limits we finally obtain

$$\rho_m(t_1, t_2, \dots, t_m) = \pi^{-m} \text{Pf} \begin{pmatrix} \frac{1}{4} \frac{\sinh \frac{t_i - t_j}{2}}{\cosh^2 \frac{t_i - t_j}{2}} & \frac{1}{2} \text{sech} \frac{t_i - t_j}{2} \\ -\frac{1}{2} \text{sech} \frac{t_i - t_j}{2} & 2 \arctan \tanh \frac{t_i - t_j}{4} - \frac{1}{2} \pi \text{sgn}(t_i - t_j) \end{pmatrix}, \quad (52)$$

with the convention that $\text{sgn}(0) = 0$.

7) GLAUBER DYNAMICS FOR THE ISING MODEL ON A HALF LINE AND MAPPING TO THE 2d-DIFFUSION EQUATION

We consider a semi-infinite Ising spin chain, whose configuration at time t is given by $\{\sigma_i(t)\}_{i \geq 0}$, with $\sigma_i(t) = \pm 1$. Initially, the system is in a random initial configuration where $\sigma_i(0) = \pm 1$ with equal probability $1/2$ and, at subsequent time, the system evolves according to the Glauber dynamics. Within each infinitesimal time interval Δt , every spin is updated according to

$$\sigma_i(t + \Delta t) = \begin{cases} \sigma_i(t), & \text{with proba. } 1 - 2\Delta t, \\ \sigma_{i-1}(t), & \text{with proba. } \Delta t, \\ \sigma_{i+1}(t), & \text{with proba. } \Delta t, \end{cases} \quad (53)$$

while the site at the $\sigma_0(t)$ evolves via $\sigma_0(t + \Delta t) = \sigma_0(t)$ with probability $1 - \Delta t$ and $\sigma_0(t) = \sigma_1(t)$ with probability Δt .

Derrida et al. in [17, 18] noticed that if one traces back in time the value of the spin i at time t , one obtains a random walk that connects the site i at time t through its various ancestors to a particular site i_0 at initial time $t = 0$ (see Fig. 1). And consequently $\sigma_i(t) = \sigma_{i_0}(0)$. It is important to notice that, in this mapping, the arrow of time has to be reversed (see Fig. 1). Then, for instance, to compare the values of the spin $\sigma_i(t)$ and $\sigma_j(t)$ at two different sites i and j , one considers two random walkers starting at site i and j . After time t , two different situations may then have occurred: (i) either the two random walkers have “coalesced” into a single walker that “ends up” on site k_0 , and consequently $\sigma_i(t) = \sigma_j(t) = \sigma_{k_0}(0)$, (ii) or the random walkers have not met each other and therefore $\sigma_i(t) = \sigma_{i_0}(t)$ while $\sigma_j(t) = \sigma_{j_0}(t)$ with $i_0 \neq j_0$. Using this mapping to coalescing random walkers, the authors of [17, 18] computed the persistence probability $p_{\text{Ising}}(t)$ for the Ising chain and found that, for large $t \gg 1$, $p_{\text{Ising}}(t) \sim t^{-\theta_{\text{Ising}}}$ with $\theta_{\text{Ising}} = 3/8$. They further generalized this result to the q -states Potts model ($q = 2$ corresponding to the Ising model) and found that the persistence probability $p_{\text{Potts}}(t)$ decays as [17, 18]

$$p_{\text{Potts}}(t) \sim t^{-\hat{\theta}(q)} \quad , \quad \hat{\theta}(q) = -\frac{1}{8} + \frac{2}{\pi^2} \left[\cos^{-1} \frac{2-q}{\sqrt{2q}} \right]^2. \quad (54)$$

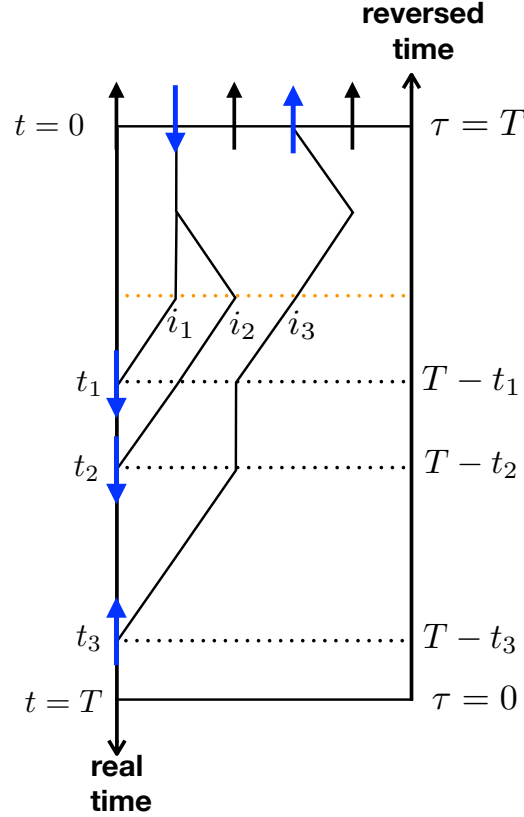


FIG. 2. Illustration of the mapping to coalescing random walks to compute the multi-time correlation functions of the spin at the origin at $N = 3$ different times t_1, t_2 and t_3 . In this specific realisation of the Glauber dynamics with a given initial configuration at $t = 0$, one reads immediately $\sigma_0(t_1)\sigma_0(t_2)\sigma_0(t_3) = (\sigma_1(0))^2 \sigma_4(0)$. Note that, at all time, $i_1 \leq i_2 \leq i_3$.

Remarkably, the expression for $\hat{\theta}(q)$ in (54) bears strong similarities with the function $\psi(s)$, associate to the $2d$ -diffusing field given in the text in Eq. (19). Below we show that this is not just a coincidence: indeed we exhibit a mapping between the (semi-infinite) Ising-chain with Glauber dynamics and the $2d$ -diffusion equation with random initial conditions. Using the formulation of the Glauber dynamics (53) in terms of coalescing random walks, that the multi-time correlation functions of σ_0 are given by a Pfaffian, as given in Eq. (20) in the text, for $t_1 < t_2 < \dots < t_{2m}$

$$\langle \sigma_0(t_1) \dots \sigma_0(t_{2m}) \rangle \sim \text{Pf}(A), \quad a_{i,j} = \text{sgn}(j-i) \langle \sigma_0(t_i) \sigma_0(t_j) \rangle \quad (55)$$

with $a_{i,j}$ as given in Eq. (21) in the text for $t_i, t_j \gg 1$. From these identities for any correlation function in Eq. (20) in the text and in Eq. (55) above, we conclude that $\text{sgn}(X(t))$ for the $2d$ -diffusion equation and $\sigma_0(t)$ in the semi-infinite Ising chain with Glauber dynamics are actually exactly the same process in the large time limit. Therefore we conclude that their persistence properties do coincide and therefore $b = 3/16$, as announced in Eq. (4) in the text.

To compute these multiple-time correlation functions $\langle \sigma_0(t_1) \dots \sigma_0(t_N) \rangle$ we again use the mapping of the $T = 0$ Glauber dynamics to coalescing random walks (see Fig. 2). A remarkable simplification occurs when one considers the site at the boundary $i = 0$ of the semi-infinite [17, 18]. Indeed, when N walkers start from the origin at times $T - t_N < T - t_{N-1} < \dots < T - t_1$ the positions of the path at any time t remain always in the same order, i.e. $i_1 \leq i_2 \leq i_N$, since when a walker is at the origin, it can only hop to the right (see Fig. 2). As we will see, the main tool for computing the multi-time correlation functions is the probability $c_{i,j}$ that two walkers, starting at $T - t_i > T - t_j$, with $i < j$, do not meet up to time T . This probability $c_{i,j}$ is of course independent of T and reads, for $t_j > t_i \gg 1$ with t_i/t_j fixed, [17, 18]

$$c_{i,j} \simeq \frac{4}{\pi} \tan^{-1} \sqrt{\frac{t_j}{t_i}} - 1, \quad \text{for } t_j > t_i \gg 1. \quad (56)$$

Similarly, if $1 \ll t_j < t_i$, one finds [17, 18]

$$c_{i,j} = -c_{j,i}, \text{ for } t_i > t_j \gg 1. \quad (57)$$

In Ref. [17, 18], this result (56) was generalized to compute the propability $c_{j_1, j_2, \dots, j_{2k}}^{(k)}$ that no pair of random walks (among the $2k$ random walkers) with labels j_1, j_2, \dots, j_{2k} meets up to time T . It turns out that it can be written as a Pfaffian,

$$c_{j_1, j_2, \dots, j_{2k}}^{(k)} = \text{Pf} \{c_{i,j}\}_{i,j=j_1, \dots, j_{2k}} = \frac{1}{k! 2^k} \sum_{\sigma \in \mathcal{S}_{2k}} \epsilon(\sigma) c_{j_{\sigma(1)}, j_{\sigma(2)}} \cdots c_{j_{\sigma(2k-1)}, j_{\sigma(2k)}}, \quad (58)$$

where the sum is over the group of permutations \mathcal{S}_{2k} of the indices $\{j_1, j_2, \dots, j_{2k}\}$ and $c_{i,j}$ is the anti-symmetric matrix defined in Eqs. (56) and (57).

Let us start with the two-point correlation function $\langle \sigma_0(t_i) \sigma_0(t_j) \rangle$ which can easily be computed using the mapping to coalescing random walks (see Fig. 2). If, after time t the two paths, starting at $t - t_i$ and $t - t_j$ have not coalesced, which happens with probability $c_{i,j}$ then $\sigma_0(t_i) \sigma_0(t_j) = \sigma_{j_0}(0) \sigma_{k_0}(0)$ for some $j_0 \neq k_0$. Thus, after averaging over the initial conditions $\sigma_{j_0}(0) = \pm 1$, as well as $\sigma_{k_0}(0) = \pm 1$, with equal probability, we see that such non-coalescing paths give a vanishing contribution to the two-point correlation function $\langle \sigma_0(t_i) \sigma_0(t_j) \rangle$. On the contrary, if the two paths have coalesced, which happens with probability $1 - c_{i,j}$, then there exists a site j_0 such that $\sigma_0(t_i) \sigma_0(t_j) = (\sigma_{j_0}(0))^2 = 1$, which remains 1 after averaging over the initial condition $\sigma_{j_0}(0) = \pm 1$. Hence, such coalescing paths give a contribution $1 - c_{i,j}$ to the two-time correlation function and therefore, for $t_j > t_i \gg 1$ with t_i/t_j fixed, one has

$$\langle \sigma_0(t_i) \sigma_0(t_j) \rangle = 1 - c_{i,j} \simeq 2 - \frac{4}{\pi} \tan^{-1} \sqrt{\frac{t_j}{t_i}} = \frac{4}{\pi} \tan^{-1} \sqrt{\frac{t_i}{t_j}}, \quad (59)$$

where, in the last equality, we have used $\tan^{-1}(x) = \pi/2 - \tan^{-1}(1/x)$. Using the identity

$$\tan^{-1}(x) = \frac{1}{2} \sin^{-1} \left(\frac{2x}{1+x^2} \right), \text{ for } 0 < x < 1, \quad (60)$$

we obtain from (59) that the two-time correlation function $\langle \sigma_0(t_i) \sigma_0(t_j) \rangle$ reads, for $t_j > t_i \gg 1$

$$\langle \sigma_0(t_i) \sigma_0(t_j) \rangle \simeq \frac{2}{\pi} \sin^{-1} \left(\frac{2\sqrt{t_i t_j}}{t_i + t_j} \right). \quad (61)$$

Similarly, using the relation in (57), we obtain, for $t_i, t_j \gg 1$,

$$\langle \sigma_0(t_i) \sigma_0(t_j) \rangle \simeq \text{sign}(j - i) \frac{2}{\pi} \sin^{-1} \left(\frac{2\sqrt{t_i t_j}}{t_i + t_j} \right), \quad (62)$$

which thus coincides precisely with the 2-point correlation of $\text{sgn}(X(t))$ given in Eq. (49), with $X(t)$ the normalised $2d$ -diffusing field.

We now prove relation (55) for any $m \geq 1$. Let us denote $T < t_{2m} < \dots < t_1$ and use the mapping to coalescing random walks in \mathbb{Z}_+ starting from the origin at times $T - t_1 > T - t_2 > \dots > T - t_{2m}$ (see Fig. 2). As explained in the case $m = 1$, if two of the random walks meet before time T , the corresponding spins have the same ancestor and therefore the same value. On the contrary, if they do not meet, the values of the spins take independently different values ± 1 with probabilities $1/2$. Assume then that the RWs end up at the points $x_1 \leq x_2 \leq \dots \leq x_{2m}$. For every partition $\mathbf{m} = (n_1, n_2, \dots, n_{m'})$ of $2m = n_1 + n_2 + \dots + n_{m'}$ we introduce the event $\mathcal{A}_{\mathbf{m}}$ consisting of paths satisfying $\mathcal{A}_{\mathbf{m}} = \{\text{configurations of coalescing RW with } x_1 = \dots = x_{n_1} < x_{n_1+1} = \dots = x_{n_1+n_2} < \dots < x_{2n-n_{m'}+1} = \dots = x_{2m}\}$. (63)

We denote by $y_1, y_2, \dots, y_{m'}$ the distinct points among x_1, x_2, \dots, x_{2m} (see Fig. 3). It is useful to introduce $\Delta_1, \dots, \Delta_{m'}$ which are the corresponding intervals of indices, i.e.,

$$\Delta_1 = [1, n_1], \Delta_\ell = [n_1 + \dots + n_{\ell-1} + 1, n_1 + \dots + n_\ell] \text{ for } \ell \geq 1, \quad (64)$$

as well as $N_1 = 1, N_2 = n_1 + 1, \dots, N_{m'} = n_1 + \dots + n_{m'-1} + 1$ which are the left endpoints of these intervals $\Delta_1, \Delta_2, \dots, \Delta_{m'}$. The only configurations that will contribute to the expectation in the l.h.s. of (55) are such that all n_i are even. Indeed,

$$\mathbb{E} [\sigma_0(t_1) \sigma_0(t_2) \cdots \sigma_0(t_{2m}) | \mathcal{A}_{\mathbf{m}}] = \mathbb{E} [\sigma_{x_1}(0) \sigma_{x_2}(0) \cdots \sigma_{x_{2m}}(0) | \mathcal{A}_{\mathbf{m}}] = \text{Pr} [\mathcal{A}_{\mathbf{m}}] \prod_{j=1}^{m'} \mathbb{E}_0 [\sigma_{y_j}^{n_j}(0)] = \text{Pr} [\mathcal{A}_{\mathbf{m}}] \mathbb{I}_{2|n_1, \dots, n_{m'}}, \quad (65)$$

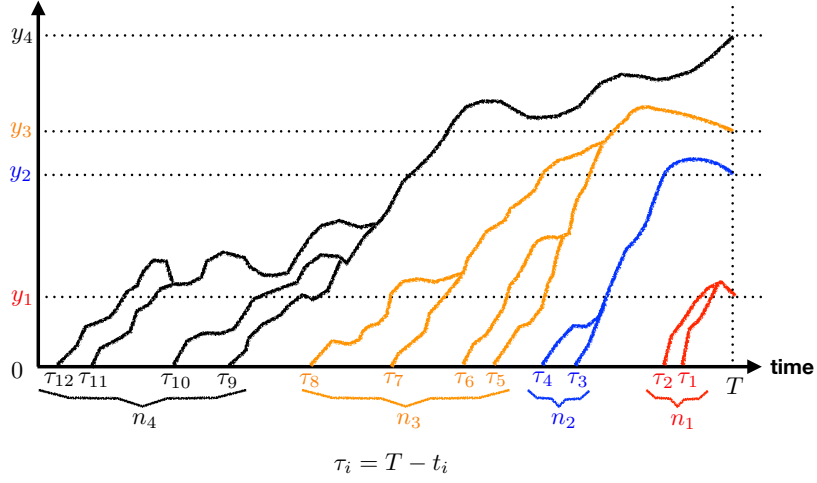


FIG. 3. Sample paths configuration for $2m = 12$ random walkers ending up in $m' = 4$ distinct points $y_1 < y_2 < y_3 < y_4$ and coalesced into 4 groups of sizes $n_1 = 2, n_2 = 2, n_3 = 4, n_4 = 4$. This configuration belongs to the set $\mathcal{A}_{\mathbf{m}}$ for partition $\mathbf{m} = (2, 2, 4, 4)$. The partition \mathbf{m} complies with non-intersection events $c_{j_1, j_2, \dots, j_{2k}}^{(k)}$ such that all $(j_1, j_2, \dots, j_{2k})$ belong to different intervals chosen from $\Delta_1 = [1, 2], \Delta_2 = [3, 4], \Delta_3 = [5, 8], \Delta_4 = [9, 12]$. For example, $c_{1,3}^{(2)}, c_{4,11}^{(2)}, c_{1,3,5,9}^{(4)}$, etc.

where $\mathbb{I}_{2|n_1, \dots, n_{m'}}$ imposes that all the n_i 's are even. In Eq. (65), we have used the independence between the spins at time zero and denoted by \mathbb{E}_0 the average over the initial condition. Therefore, one can now rewrite multi-time spin autocorrelation function as follows

$$\langle \sigma_0(t_1) \sigma_0(t_2) \cdots \sigma_0(t_{2m}) \rangle = \sum_{\mathbf{m}} \mathbb{E} [\sigma_0(t_1) \sigma_0(t_2) \cdots \sigma_0(t_{2m}) | \mathcal{A}_{\mathbf{m}}] = \sum_{\mathbf{m} | n_i \in 2\mathbb{N}} \Pr[\mathcal{A}_{\mathbf{m}}]. \quad (66)$$

The next step is to expand the Pfaffian in the right hand side of (55) by using the decomposition formula valid for any anti-symmetric matrices A, B of the same size $2m \times 2m$ (see e.g. [19])

$$\text{Pf}(A + B) = \sum_{\mathbf{J} \subset \{1, 2, \dots, 2m\}} (-1)^{\sum j_k - |\mathbf{J}|/2} \text{Pf}(A|_{\mathbf{J}}) \text{Pf}(B|_{\mathbf{J}^c}), \quad (67)$$

where the sum runs over all even sized subsets $\mathbf{J} = (j_1, j_2, \dots, j_{|\mathbf{J}|})$ with $|\mathbf{J}|$ elements (\mathbf{J}^c denoting its complementary in the set $\{1, 2, \dots, 2m\}$) and $A|_{\mathbf{J}}$ denotes a minor of the matrix A containing rows and columns with indices taken from \mathbf{J} (and the Pfaffian of the empty matrix is taken to have value 1). By specializing this formula (67) to the matrices $A = \{\text{sgn}(j - i)\}_{i,j=1}^{2m}$ (with the convention that $\text{sgn}(0) = 0$) and $B = -\{c_{i,j}\}_{i,j=1}^{2m}$, with $c_{i,j}$ as defined in Eqs. (56) and (57) we get

$$\text{Pf}(\text{sgn}(j - i) - c_{i,j}) = \sum_{k=0}^m \sum_{1 \leq j_1 < j_2 < \dots < j_{2k} \leq 2m} (-1)^{j_1 + j_2 + \dots + j_{2k}} \text{Pf}\{c_{i,j}\}_{j_1, j_2, \dots, j_{2k}} \quad (68)$$

$$= \sum_{k=0}^m \sum_{1 \leq j_1 < j_2 < \dots < j_{2k} \leq 2m} (-1)^{j_1 + j_2 + \dots + j_{2k}} c_{j_1, j_2, \dots, j_{2k}}^{(k)} \quad (69)$$

where, in the second line, we have used that the Pfaffian appearing in the first line has a probabilistic interpretation in terms of non-intersecting paths [see Eq. (58)]. Note that when applying the formula (67) to the aforementioned matrices A and B we have used that $\text{Pf}(A|_{\mathbf{J}}) = 1$ for all even sized subset \mathbf{J} as well as $\text{Pf}(B|_{\mathbf{J}^c}) = (-1)^{|\mathbf{J}^c|} \text{Pf}(-B|_{\mathbf{J}^c})$.

We recall that the event $\mathcal{A}_{\mathbf{m}}$ with partition $\mathbf{m} = (n_1, n_2, \dots, n_{m'})$ of $2m = n_1 + \dots + n_{m'}$ consists of paths that meet at m' distinct points $y_1, y_2, \dots, y_{m'}$ at time T [see Eq. (63), Fig. 3]. Therefore the probability of this event is the sum of all probabilities $c_{j_1, j_2, \dots, j_{2k}}^{(k)}$ with the constraint that the indices $j_1 < j_2 < \dots < j_{2k}$ belong to different intervals Δ_ℓ of the partition (64). In this case we will say that \mathbf{J} comply with \mathbf{m} and write $\mathbf{J} \sim \mathbf{m}$. We can thus rewrite the right hand side of Eq. (69) as

$$\text{Pf}(\text{sgn}(j - i) - c_{i,j}) = \sum_{\mathbf{m}} \alpha_{\mathbf{m}} \Pr[\mathcal{A}_{\mathbf{m}}], \quad \alpha_{\mathbf{m}} = \sum_{k=0}^m \sum_{\mathbf{J} \sim \mathbf{m}} (-1)^{j_1 + j_2 + \dots + j_{2k}}. \quad (70)$$

If $\mathbf{J} \sim \mathbf{m}$ then the index j_1 belongs to an interval Δ_{p_1} , for some p_1 . Therefore, all the subsets $\mathbf{J}' = \{j'_1, j_2, \dots, j_{2k}\}$ which differ from \mathbf{J} only by the first index j'_1 such that $j'_1 \in \Delta_{p_1}$ also complies with the partition \mathbf{m} . By summing over such groups of subsets, i.e. over j_1 , gives

$$\Sigma_{p_1} := \sum_{j_1 \in \Delta_{p_1}} (-1)^{j_1} = \begin{cases} 0, & \text{if } n_{p_1} \text{ is even,} \\ (-1)^{N_{p_1}}, & \text{if } n_{p_1} \text{ is odd.} \end{cases} \quad (71)$$

Grouping now the subsets \mathbf{J} that differ by two indices, and then by three indices, etc, we finally get

$$\alpha_{\mathbf{m}} = \sum_{k=0}^{\lfloor m'/2 \rfloor} \sum_{1 \leq p_1 < p_2 < \dots < p_{2k} \leq m'} \prod_{\ell=1}^{2k} \Sigma_{p_\ell}, \quad (72)$$

where the empty product corresponding to $k = 0$ is assumed to be equal to 1 and where $[x]$ means the integer part of x . The identity (72) means that the coefficient $\alpha_{\mathbf{m}}$ is equal to the sum of all even products of variables $\Sigma_1, \Sigma_2, \dots, \Sigma_{m'}$. This can be easily rewritten as

$$\alpha_{\mathbf{m}} = \frac{1}{2} \left[\prod_{\ell=1}^{m'} (1 + \Sigma_\ell) + \prod_{\ell=1}^{m'} (1 - \Sigma_\ell) \right]. \quad (73)$$

If for some \mathbf{m} all n_i 's are even then for any ℓ we have $\Sigma_\ell = 0$ and consequently $\alpha_{\mathbf{m}} = 1$. For partitions containing odd intervals we denote ℓ_1 and ℓ_2 to be numbers of the first two intervals of odd length. Then N_{ℓ_1} is odd and $\Sigma_{\ell_1} = -1$ and the first term in above expression vanishes. But one can also see that N_{ℓ_2} is even and $\Sigma_{\ell_2} = 1$ which implies that the second term vanishes as well. Therefore we see that $\alpha_{\mathbf{m}} \neq 0$ if and only if all n_i are even. And hence the Pfaffian in the left hand side of Eq. (70) can be written as

$$\text{Pf}(\text{sgn}(j-i) - c_{i,j}) = \sum_{\mathbf{m} | n_i \in 2\mathbb{N}} \text{Pr}[\mathcal{A}_{\mathbf{m}}] = \langle \sigma_0(t_1) \sigma_0(t_2) \dots \sigma_0(t_{2m}) \rangle, \quad (74)$$

where, in the last equality, we have used Eq. (66). Finally, using the asymptotic behaviour of $1 - c_{i,j}$ for $t_i < t_j$ in Eqs. (59) and (61), we obtain the result announced in Eq. (22) in the text.

-
- [1] M. Krishnapur, *From random matrices to random analytic functions*, Ann. Probab. **37**, 314 (2009).
 - [2] J. B. Hough, M. Krishnapur, Y. Peres, B. Viràg, *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*, University Lecture Series **51**, 2009.
 - [3] P. J. Forrester, *The limiting Kac random polynomial and truncated random orthogonal matrices*, J. Stat. Mech. **12**, P12018 (2010).
 - [4] S. Matsumoto, T. Shirai, *Correlation functions for zeros of a Gaussian power series and Pfaffians*, Electron. J. Probab. **18**, 1 (2013).
 - [5] P. J. Forrester, J. R. Ipsen, *A generalisation of the relation between zeros of the complex Kac polynomial and eigenvalues of truncated unitary matrices*, [arxiv.org:1807.06743](https://arxiv.org/abs/1807.06743).
 - [6] M. Gebert, M. Poplavskiy, *On pure complex spectrum for truncations of random orthogonal matrices and Kac polynomials*, in preparation, 2018.
 - [7] B. A. Khoruzhenko, H.-J. Sommers, K. Zyczkowski, *Truncations of random orthogonal matrices*, Phys. Rev. E **82**, 040106 (2010).
 - [8] A. Borodin, C. D. Sinclair, *The Ginibre Ensemble of Real Random Matrices and its Scaling Limits*, Commun. Math. Phys. **291**, 177 (2009).
 - [9] C. D. Sinclair, *Averages over Ginibre's Ensemble of Random Real Matrices*, Int. Math. Res. Not., rnm015 (2007).
 - [10] NIST Digital Library of Mathematical Functions, <https://dlmf.nist.gov/26.8#i>
 - [11] L. C. G. del Molino, K. Pakdaman, J. Touboul, G. Wainrib, *The Real Ginibre Ensemble with $k = O(n)$ Real Eigenvalues*, J. Stat. Phys. **163**, 303 (2016).
 - [12] P. J. Forrester, S. Kumar, *The probability that all eigenvalues are real for products of truncated real orthogonal random matrices*, J. Theor. Probab., 1-16 (2017).
 - [13] J. M. Hammersley, *The zeros of a random polynomial*, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Vol. 2, (1972).
 - [14] G. Schehr, S. N. Majumdar, *Statistics of the Number of Zero Crossings: from Random Polynomials to Diffusion Equation*, Phys. Rev. Lett. **99**, 060603 (2007).

- [15] G. Schehr, S. N. Majumdar, *Real Roots of Random Polynomials and Zero Crossing Properties of Diffusion Equation*, J. Stat. Phys. **132**, 235 (2008).
- [16] M. Ishikawa, H. Kawamuko, S. Okada, *A Pfaffian-Hafnian analogue of Borchardts identity*, Electron. J. Combin. **12** (2005).
- [17] B. Derrida, V. Hakim, V. Pasquier, *Exact first-passage exponents of 1d domain growth - relation to a reaction-diffusion model*, Phys. Rev. Lett. **75**, 751 (1995).
- [18] B. Derrida, V. Hakim, V. Pasquier, *Exact Exponent for the Number of Persistent Spins in the Zero-Temperature Dynamics of the One-Dimensional Potts Model*, J. Stat. Phys. **85**, 763 (1996).
- [19] R. Tribe, O. Zaboronski, *Pfaffian formulae for one dimensional coalescing and annihilating systems*, Electron. J. Probab. **16**, 2080 (2011).